Tatonnement Beyond Gross Substitutes? Gradient Descent to the Rescue

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Abstract

We identify a class of economies for which tatonnement is equivalent to gradient descent. This is the class of economies for which there is a convex potential function whose gradient is always equal to the negative of the excess demand. Among other consequences, we show that a discrete version of tatonnement converges to the equilibrium for the following economies of complementary goods.

i. Fisher economies in which all buyers have complementary CES utilities, with a linear rate of convergence. (In Fisher economies all agents are either buyers or sellers of non-numeraire goods, but not both.)

This shows that tatonnement converges for the entire range of Fisher economies when buyers have complementary CES utilities, in contrast to prior work, which could analyze only the substitutes range, together with a small portion of the complementary range.

ii. Fisher economies in which all buyers have Leontief utilities, with an $O(1/t)$ rate of convergence.

Keywords: Market; Equilibria; Gradient Descent; Tatonnement

1. Introduction

Two central questions in general equilibrium theory are whether equilibria exist and if so how to compute them. The issue of existence was settled for a very general setting in 1954 by Arrow and Debreu [3]. The formal study of this topic began with the introduction of an equilibrium model by Walras in 1874 [64], along with an intuitive, simple, distributed price update process which he named tatonnement.[1]

Tatonnement is broadly defined as follows: if the demand for a good exceeds its supply, increase its price, and conversely, decrease its price when the demand is smaller. Classically, tatonnement has been thought of as a *continuous* process, with price adjustments and demand responses happening continuously. A computer science approach is to consider updates at *discrete* time intervals and to bound the number required (though discrete updates were also considered in the economics literature as early as the 60s [63]).

A crucial issue is whether economies reach and remain at equilibria, and do so in a reasonable time. The classic approach to this question is the study of stability, both local and global. This asks, given a price adjustment scheme, such as tatonnement, in the form of a differential equation, whether the prices converge to an equilibrium (a) when starting at a point near enough to the equilibrium (local stability) or (b) when starting at (almost) any set of prices (global stability).

An early positive result, due to Arrow, Block and Hurwitz [2], showed that a continuous version of tatonnement is globally stable for economies satisfying the weak gross substitutes (WGS) property, namely that increasing the price of one good does not decrease the demand for any other good. However, the hope that tatonnement would converge for all economies was dashed by Scarf [60], who showed an example of an economy in which tatonnement exhibits cyclic behavior. Thus one can hope to show that tatonnement converges at all, let alone quickly, only for specific classes of economies.

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[1] Tatonnement is distributed in the sense that each price can be updated independently of the current updates to other prices.

This then led to the consideration of other price update procedures, most notably the methods due to Scarf [61] and Kuhn [53], and combinations of these [43, 54, 48]. These methods have high information demands, which makes them somewhat implausible as models of real-world price-updating, but as shown by Saari and Simon [59], this is unavoidable in general. Another approach has been to identify classes of economies for which tatonnement is globally stable. These include economies having weighted diagonal dominance [30, 31], satisfying the strong law of demand [49], and having aggregate substitution effects [50].

More recently, these same issues have been examined from the perspective of computational complexity. This work can be organized in terms of two broad questions. The first asks whether equilibria, or at least good approximations, can be reached in a reasonable time, which is reformulated as whether they can be reached in polynomial time[2]. The answer appears to be "no" in general: the problem is NP-hard for economies of indivisible goods and it is PPAD-hard for economies of divisible goods [23, 14, 13]. This led to much work on identifying classes of economies[3] according to whether there are polynomial time algorithms for computing equilibria, either exactly or approximately [29, 56, 26, 32, 21, 38, 28]. The second broad question is whether there are simple, "natural" processes that provide these convergence guarantees, such as tatonnement [24, 18] and proportional response [65, 66, 6, 19].

We note that analogous questions have been asked about Nash equilibria. Here too, in general their computation is PPAD-hard [25, 12]. Also, as shown by Hart and Mansour [42], there are no "natural dynamics" that reach a Nash equilibrium quickly in general. Natural dynamics have been shown to converge quickly only for limited classes of games [37, 20].

In the same spirit, our goal is to identify broad classes of economies for which tatonnement converges quickly toward an equilibrium. Cole and Fleischer [24] showed this for a discrete version of tatonnement for a class of economies satisfying the weak gross substitutes property. The current paper is particularly concerned with economies that exhibit *complementarity*, such as the *Constant Elasticity of Substitution* (CES) utilities and *Leontief* utilities; it will focus mainly on Fisher economies, economies in which the agents can be partitioned into buyers and sellers.[4]. (See Section 2 for formal definitions.)

The existing results all rely on very strong properties of WGS economies. For example, for Fisher economies, tatonnement guarantees that the bound on the ratio of the current price to the equilibrium price always shrinks [24]. Another example, again for Fisher economies, is that the equilibrium can be reached by starting with very small prices and increasing them *monotonically* [39]. These strong properties cease to hold in the complementary regime. Therefore new techniques are needed to handle such economies.

Our contributions. We identify settings for which tatonnement amounts to another simple and natural process: *gradient descent*. Gradient descent is a family of algorithms used to minimize convex functions. It works by starting at some point and moving in the direction of the negative of the gradient. We consider the class of economies for which tatonnement is formally equivalent to performing gradient descent on a convex function. We note that while gradient descent has been viewed as a tatonnement rule in various contexts [1, 7, 10, 57], it is not immediately clear that in general, given an economy, there is a function on which gradient descent corresponds to tatonnement. Thus in order to treat tatonnement as gradient descent one has to identify a convex function corresponding to the economy in question. Accordingly, we define the class of Convex Potential Function (CPF) economies to be those economies for which there is a convex potential function whose gradient[5] is always equal to the negative of the excess demand. We show that this class contains the class of Eisenberg-Gale (EG) economies introduced by Jain and Vazirani [46].

The equivalence with gradient descent opens up the entire tool box developed to analyze gradient descent and provides a principled approach to show convergence of the tatonnement process. We show convergence

[2]Polynomial time refers to a computation that runs in a number of steps that is bounded by a polynomial function of the size in bits of the problem instance description, which in this context is the specification of the economy.

[3]In much of the computer science literature, economies are called markets.

[4]The term Fisher market was coined by the computer science community to refer to this class of economies, which were the economies defined by Fisher and experimented with using a hydraulic apparatus (see [9] for a description).

[5]More generally, the potential function need not be differentiable and the demand need not be unique, in which case the equivalence is between the subgradient of the potential function and the set of excess demand vectors.

for a discrete version of tatonnement for Fisher economies with CES and Leontief utilities, by proving certain structural properties of the corresponding convex functions for these economies.

While one would expect these results to extend to the case of continuous updates, it is not even immediately evident how to define a continuous version of tatonnement based on the type of gradient descent we consider. Nonetheless, for a large class of CPF economies, we show that a continuous version of tatonnement converges to an equilibrium. Interesting, the tatonnement rules we consider are quite general, including rules that are linear in the price, unlike most other work on global convergence.

We now summarize the main results in the paper.

i. The class of Eisenberg-Gale (EG) economies comprises all Fisher economies for which the equilibrium allocation is captured by a certain type of convex program called the Eisenberg-Gale-type (EG-type) convex program. We show that EG economies are CPF economies by explicitly constructing a convex potential function (Theorem 3.1).

ii. For Fisher economies with Leontief utilities, we show a fairly fast rate of convergence for a discrete version of the process, namely, the number of time steps required to reduce the distance from the equilibrium to an ϵ fraction of its initial value, as measured by the potential function, is $O(1/\epsilon)$ (Theorem 4.1). [6]

We also show that, in the worst case, tatonnement uses $\Omega(1/\sqrt{\epsilon})$ iterations. Consequently, the linear convergence bounds achieved for complementary CES utilities (see below) cannot extend to Leontief utilities.

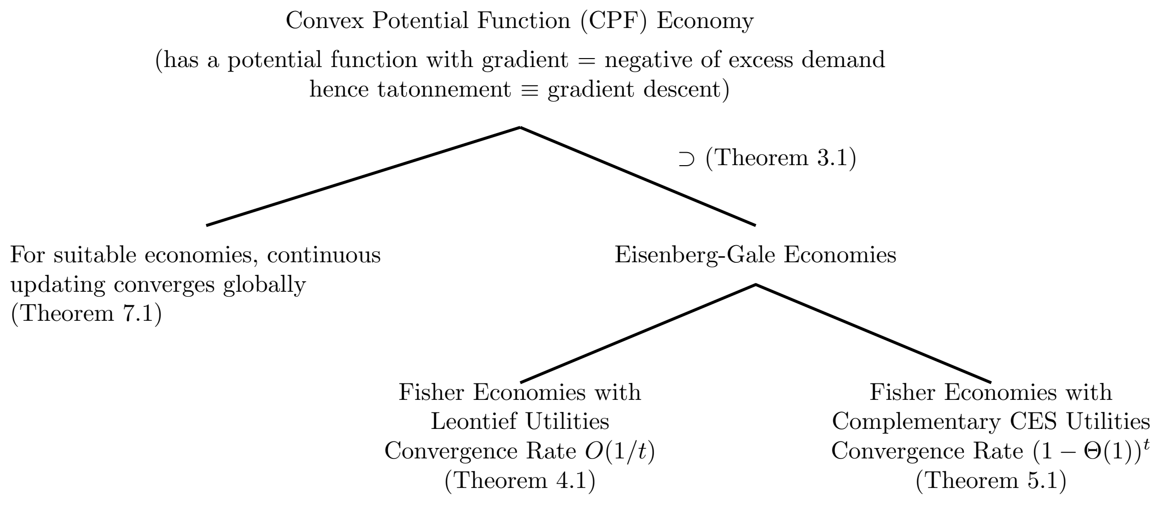
iii. For Fisher economies with complementary CES utilities we show a linear convergence, i.e., the number of time steps required to reduce the distance from the equilibrium to an ϵ fraction of its initial value, again as measured by the potential function, is $O(\log(1/\epsilon))$ (Theorem 5.1).

In addition, we show that this analysis extends to CES utilities that are substitutes, providing an alternate analysis for some of the results in [24].

iv. We show that a family of continuous versions of tatonnement process converges from essentially any starting prices to an equilibrium for a large subclass of CPF economies.

Figures 1 summarizes our results and their relationships. Figure 2 at the end of Section 3 shows the structure of the analyses for discrete tatonnement updates.

Figure 1: Classes of Economies



Related Work. The stability of the tatonnement process has been considered to be one of the most fundamental issues in general equilibrium theory. Hahn [41] provides a thorough survey on the topic, and the textbook of Mas-Colell, Whinston and Green [55] contains a good summary of the classic results.

[6] The $O()$ hides economy dependent parameters.

The longstanding interpretation of tatonnement is that it is a method used by an auctioneer for iteratively updating prices, followed by trading at the equilibrium prices once they are reached. If trading is allowed as the price updating occurs, this is called a *non-tatonnement* process. Thus tatonnement is not an "in-market" process, and nor is it for any other price update rule for a static model such as the Arrow-Debreu model. Hermann and Kahn [44] stated that "it is certainly desirable to find a mechanism that mimics the actual market adjustment process". Fisher [35], discussing the classic auctioneer interpretation of tatonnement, states: "such a model of price adjustment ... describes nobody's actual behavior". The main point of [35], however, was to give an alternate and more plausible basis for tatonnement. More recently, Cole and Fleischer [24] sought to provide another self-contained basis for the tatonnement price-update by introducing the Ongoing Market model, in which tatonnement and other price update processes can naturally be viewed as in-market processes (these are non-tatonnement processes). The continued interest in the plausibility of tatonnement is also reflected in some experiments by Hirota [45], which showed the predictive accuracy of tatonnement in a non-equilibrium trade setting.

Equilibria in economies with complementary goods have been considered by Arrow and Hahn [4], who introduced a general notion of Diagonal Dominance and showed it implies a unique equilibrium and local stability (though it is not clear whether it implies global stability). Subsequently, Dohtani [30] introduced another condition of a similar flavor which did imply global stability. However, the relation of each of these conditions to the classes we consider is not immediately obvious.[7]

Gradient and subgradient descent have been viewed as tatonnement in various contexts, including combinatorial auctions [7], power markets [57], multi-agent scheduling [1], and others [10, p. 495].

The use of differential inclusions[8], instead of differential equations that require unique demands, was initiated by Champsaur et al. [11]; we will also be using differential inclusions for our continuous analysis.

In recent years, discrete versions of tatonnement have received increased attention. Codenotti et al. [22] considered a tatonnement-like process that required some coordination among different goods and showed polynomial time convergence for WGS economies. Cole and Fleischer [24] were the first to establish fast convergence for a truly distributed discrete version of tatonnement, once again for a class of WGS economies. Cheung, Cole and Rastogi [18] extended this result slightly beyond WGS economies, to CES utilities for a limited range of parameters.[9] In comparison, our results cover the entire range of parameters for CES utilities. Fleischer et al. [36] also consider price dynamics that are similar to tatonnement but they also need coordination and further, their results concern the average price throughout the process rather than convergence of the sequence.

In a similar spirit to this paper, Birnbaum, Devanur and Xiao [6] considered another distributed process called the Proportional Response (PR) dynamics for the linear utilities case, showed its equivalence to gradient descent with KL-divergence for a different convex function and obtained convergence rates for the process. PR dynamics work in the space of offers rather than the space of prices, which is why the corresponding convex function is different. For linear utilities, PR dynamics are more appropriate than tatonnement, since the demand function is not continuous. [6] proved a certain convergence result (Theorem 2.1) which we use in this paper to show convergence for the case of Leontief utilities.

EG economies were defined by Jain and Vazirani [46], after observing that many economies in the Fisher model had similar convex programs that captured the equilibrium. The following is a brief list of such economies: Eisenberg and Gale [34] gave a convex program for the linear utilities case, generalized by Eisenberg [33] to the case of homothetic utilities, Jain et al. [47] for homothetic utilities with production, and Kelly and Vazirani [52] for certain network-flow economies. Jain and Vazirani [46] showed many algorithmic and structural properties of such economies.

[7]We can show that Diagonal Dominance holds for Fisher CES economies with $\rho > -1$ (see Section 5 for the definitions of these terms), by setting $h_i(p_i) = p_i$ (using the notation in [30]).

[8]"Differential inclusion" is sometimes called "differential correspondence".

[9]CES utilities are parameterized by an exponent, ρ. When $0 < \rho \leq 1$ the economy is WGS, and when $\rho < 0$ the goods are complementary. [18] analyzed the range $-1 < \rho \leq 0$.

2. Preliminaries

In the next three subsections we review, in turn, some standard definitions relating to economies, the definition of Eisenberg-Gale economies, and some known results about gradient descent. Then, in Section 2.4, we turn to the definitions of the new classes of economies we shall be analyzing in this paper.

2.1. Definitions Relating to Economies

Definition 2.1. *An* Exchange economy *has m divisible goods and n agents. Each agent i has a utility function $u_i : \mathbb{R}^m_+ \to \mathbb{R}$ that specifies the agent's utility for a given bundle of goods. Each agent i has an initial endowment of e_{ij} amount of good j. The supply of good j, $w_j := \sum_i e_{ij}$ is the total endowment of good j among all the agents. Without loss of generality we choose the units of measurement such that the supplies are all 1. Suppose we assign a price p_j to each good j, then a (possibly non-unique)* demand *of agent i is a bundle of goods $(x_{i1}, x_{i2}, \ldots, x_{im})$ that maximizes her utility function subject to the budget constraint, namely that she does not spend more than the value of her endowment; it is a solution to the following optimization problem:*

$$\begin{aligned} \max \quad & u_i(x_{i1}, x_{i2}, \ldots, x_{im}) \\ s.t. \ \ \forall i, \quad & \sum_j p_j x_{ij} \le \sum_j p_j e_{ij}, \\ \forall\, i, j, \quad & x_{ij} \ge 0. \end{aligned}$$

A market demand $(x_1, x_2, \cdots, x_m)$ is a sum of demands of different buyers, which again can be non-unique. Market demand depends on the prices $p = (p_1, p_2, \ldots, p_m)$. Prices $p^* = (p^*_1, p^*_2, \cdots, p^*_m)$ form a market equilibrium *if the resulting market can clear: there exists a market demand at these prices such that for all j, $x_j \le w_j = 1$, and further the inequality is an equality if $p^*_j > 0$. For notational convenience, we define an* excess demand *for good j by $z_j = x_j - 1$.*

Note that if the utility function is strictly concave, then there is a unique utility maximizing bundle when the prices are all positive, so we can talk of *the* demand of an agent. If all agents have strictly concave utility functions, we can talk of *the* market demand.

It is known that equilibrium prices exist if the utility functions are all monotonic, quasi-concave and continuous.

An alternate model is the *Fisher economy* model, defined next.

Definition 2.2. *In a* Fisher economy *there is a fixed endogenous supply of each good (which is again chosen to be 1 unit). The agents have a fixed endowment of* money*, which defines their budget constraint. Let agent i have an endowment of e_i units of money. The budget constraint for agent i is $\sum_j p_j x_{ij} \le e_i$. Each agent has a utility function as in the Exchange economy, with the detail that the agent has no desire for its money, i.e. each agent seeks to spend all its money on goods. Again, prices p^* form a market equilibrium if there exists market demand at these prices that clears the market.*

The Fisher model is actually a special case of the exchange model. To see this, view the supplies of the good as being initially owned by an agent called the *seller*, and view money as another good, while the seller desires only money.

We now define some interesting subclasses of economies.

Definition 2.3. *An economy satisfies the* Weak Gross Substitutes *(WGS) property, or equivalently an economy is a WGS economy, if increasing the price of any one good does not reduce the demand for any other good. If the demand function is differentiable, then this property can be written as*

$$\frac{\partial x_j}{\partial p_{j'}} \ge 0, \quad \forall\, j \ne j'.$$

In terms of the Jacobian of the demand function, in a WGS economy all the off-diagonal entries are non-negative.

Definition 2.4. *The* Leontief *utilities are of the form* $u_i = \min_j\{x_{ij}/b_{ij}\}$*. In order to get one unit of utility one needs* b_{ij} *units of good* j*, for every good* j*.*

Thus Leontief utilities capture the case of *perfect complements.* It is easy to see that the demand for good j is

$$x_{ij} = \beta_i b_{ij}, \quad \text{where } \beta_i = \frac{e_i}{\sum_j b_{ij} p_j}. \tag{1}$$

Thus the maximum utility buyer i can obtain is

$$u_i = \frac{e_i}{\sum_j b_{ij} p_j}. \tag{2}$$

Definition 2.5. *Utilities with a* Constant Elasticity of Substitution*, CES utilities for short, are of the form*

$$u_i = (a_{i1}x_1^{\rho_i} + a_{i2}x_2^{\rho_i} + \cdots + a_{im}x_m^{\rho_i})^{1/\rho_i} \tag{3}$$

with $\rho_i \le 1$ *and* $a_{ij} \ge 0$*.*

If $0 < \rho_i \le 1$ then the goods are substitutes; the goods are complementary when $\rho_i < 0$. Leontief utilities are obtained in the limit, as $\rho \to -\infty$. The utility function obtained in the limit, as $\rho \to 0$, is called the Cobb-Douglas utility.

2.2. Eisenberg-Gale Economies

Eisenberg-Gale economies are a class of economies in which one can have a single representative consumer. Following Jain and Vazirani [46], we first define Eisenberg-Gale type convex programs.

Definition 2.6. *An* Eisenberg-Gale-type *convex program is a convex program of the form*

$$\begin{array}{lll} & \max & \sum_i e_i \log u_i(x_{i1}, x_{i2}, \ldots, x_{im}) \\ s.t. & \forall\, j, & \sum_i x_{ij} \le 1, \text{ (supply constraints)} \\ & \forall\, i, j, & x_{ij} \ge 0. \end{array}$$

The base of the log does not matter for the maximization in the convex program. However, later in the paper some calculations are simplified if we assume that the natural logarithm is intended, and so we assume this henceforth.

We note that the above program satisfies Slater's conditions for strong duality (see [8], p. 226, for example) and consequently an optimal solution to the dual problem yields the same optimizing value as the primal program.

Definition 2.7. *An* Eisenberg-Gale *(EG) economy is a Fisher economy for which the optimal solution and the (corresponding) Lagrange multipliers of the supply constraints in the above convex program are respectively equilibrium allocations and prices for the economy.*

Conversely, equilibrium allocations and prices are respectively an optimal solution and Lagrange multipliers of the supply constraints to the above convex program. Note that any strictly monotone transformation of the utility function leaves the economy unchanged, since the demand function is invariant under such transformations. Thus one may need to apply suitable monotone transformations to the utility functions in order to obtain an EG economy.

It is known that buyers with Leontief and CES utilities in the Fisher model form EG economies (to see this, it suffices to note that these utility functions are homogeneous of degree 1 and then apply Theorem 2 in [33]).

2.3. Generalized Gradient Descent

We next present a generalized version of gradient descent and a corresponding convergence result.

Definition 2.8. *For any strictly convex differentiable function h, the* Bregman divergence *with kernel h is defined as*

$$d_h(p, q) = h(p) - h(q) - \nabla h(q) \cdot (p - q). \tag{4}$$

For example, the square of the Euclidean distance is obtained as a Bregman divergence, $\|p - q\|^2 = d_h(p, q)$, if $h(p) = \frac{1}{2}\|p\|^2$. Another well-known example is the KL-divergence $d_h(p, q) = \sum_j \left(p_j \log \frac{p_j}{q_j} + q_j - p_j\right)$, which is obtained when

$$h(p) = \sum_j p_j \log p_j - p_j. \tag{5}$$

For a convex function ϕ, define the first order approximation to ϕ as follows:

$$\ell_\phi(p; q) = \phi(q) + \nabla\phi(q) \cdot (p - q) \tag{6}$$

where, for each q, $\nabla\phi(q)$ denotes an arbitrary subgradient of ϕ at q. For any given point q, $\ell_\phi(p; q)$ is a linear function in p, which can be viewed as the equation for a tangent hyperplane at the point q.

Definition 2.9. *Given a specific subgradient of $\phi(q)$ for every q, the* generalized gradient descent *w.r.t. a Bregman divergence d_h on the convex function ϕ is a sequence $p^0, p^1, \ldots, p^t \ldots$, defined inductively (for any given starting point p^0) by*

$$p^{t+1} = \arg\min_p \{\ell_\phi(p; p^t) + d_h(p, p^t)\}. \tag{7}$$

Note that if the subgradient is not unique, then each selection of choices for the subgradient values may result in a distinct sequence of prices.

For the quadratic kernel, $h(p) = \frac{1}{2}\|p\|^2$, the above update rule reduces to the usual gradient descent rule:

$$p^{t+1} = p^t - \nabla\phi(p^t).$$

If the kernel is the weighted entropy, $h(p) = \sum_j \gamma_j(p_j \log p_j - p_j)$ for some weights γ_j, the update rule is

$$p_j^{t+1} = p_j^t \cdot \exp\left(-\frac{\nabla_j \phi(p^t)}{\gamma_j}\right), \quad \text{for all } j. \tag{8}$$

Birnbaum, Devanur and Xiao [6] showed the following convergence result for this generalized gradient descent (7).

Theorem 2.1 ([6]). *Suppose that for all choices of the subgradient, the convex function ϕ and the kernel h satisfy: for all p, q,*

$$\phi(p) \leq \ell_\phi(p; q) + d_h(p, q). \tag{9}$$

Let p^* be a minimizer of ϕ. Then for all t,

$$\phi(p^t) - \phi(p^*) \leq \frac{d_h(p^*, p^0)}{t}.$$

We need a slightly more general version of this theorem where we require (9) to hold only for consecutive pairs p^t, p^{t+1} for all t, instead of requiring it for all pairs p, q. It is easy to see that their proof needs only this weaker condition, yielding the following theorem.

Theorem 2.2. *Suppose that the sequence of prices p^t obey the following condition:*

$$\phi(p^{t+1}) \leq \ell_\phi(p^{t+1}; p^t) + d_h(p^{t+1}, p^t). \tag{10}$$

Let p^* be a minimizer of ϕ. Then for all t,

$$\phi(p^t) - \phi(p^*) \leq \frac{d_h(p^*, p^0)}{t}.$$

The discrete version of the tatonnement process we consider will be equivalent to the gradient descent (7) where h is the weighted entropy function, i.e., update (8) for a suitable choice of weights γ_j. The potential function ϕ will satisfy $\nabla_j \phi = -z_j$. The continuous versions we consider are presented in Section 7.

2.4. New Definitions

Definition 2.10. *An economy is said to be a* Convex Potential Function *(CPF) economy if there is a convex potential function ϕ of the prices such that for all prices p, $\nabla\phi(p) = -z(p)$. By abuse of notation, we let $\nabla\phi$ denote the set of sub-gradients when ϕ is not differentiable*[10] *and we let $z(p)$ denote the set of excess demand vectors when the demand is not unique.*

3. EG Economies

In this section we prove the following theorem.

Theorem 3.1. *All EG economies are CPF economies.*

Proof. We give an explicit construction of a convex potential function ϕ for which $\nabla\phi(p) = -z(p)$. ϕ is actually the dual of the corresponding EG-type convex program. Recall that the EG-type convex program has variables x_{ij} for all i and j. We let X denote the set of all these variables. Also recall that the optimum solution gives the equilibrium allocation and the optimal Lagrangian multipliers of the supply constraints in the program are the equilibrium prices. The *saddle point conditions* characterize the optimal solution to a convex program and the corresponding *Lagrange multipliers*, in terms of the *Lagrangian function*, which is obtained by multiplying the supply constraints by the prices and adding them to the objective function.

$$L(X, p) := \sum_i e_i \log(u_i) - \sum_{i,j} p_j x_{ij} + p \cdot \mathbb{1},$$

on the domain $\{X, p\colon \forall i, j,\ x_{ij} \geq 0;\ \forall j,\ p_j \geq 0\}$. X^* and p^* are said to satisfy the saddle point conditions if

$$\begin{aligned} &1.\ X^* \in \arg\max_{X \geq 0} L(X, p^*) \text{ and} \\ &2.\ \ p^* \in \arg\min_{p \geq 0} L(X^*, p). \end{aligned} \tag{11}$$

We define the potential function to be the dual objective of the EG-type convex program.

$$\phi(p) := \max_{X \geq 0} L(X, p).$$

ϕ is convex by construction. Theorem 3.1 follows from Lemma 3.2, which shows that the gradient of ϕ is equal to the negative of the excess demand.

The key property of EG economies is captured by the following lemma.

[10] We assume throughout that ϕ is continuous.

Lemma 3.1. *For an EG economy, for all p, the demand set $x(p)$ is exactly equal to* $\arg\max_{X\geq 0} L(X,p)$, *whenever they are both finite.*

PROOF.
Part 1, showing $x(p) \subseteq \arg\max_{X\geq 0} L(X,p)$: We first argue that if $x(p)$ is a demand at price p then it must also maximize $L(X,p)$. In fact, we first argue it for the special case when the price and the demand form an equilibrium, denoted by p^* and $x(p^*)$. Since this is an EG economy, by its definition, the pair $(x(p^*), p^*)$ must correspond to an optimal solution of the corresponding convex program. They must therefore satisfy the corresponding saddle point conditions (11), which imply that $x(p^*) \in \arg\max_{X\geq 0} L(X,p^*)$ as desired.

This immediately shows the same for any price p and every demand $x(p)$, since the pair forms an equilibrium when the supply is equal to $x(p)$. Thus the above holds for all prices and for all demand vectors.
Part 2, $\arg\max_{X\geq 0} L(X,p) \subseteq x(p)$: The argument is similar to Part 1. Consider any p and an X that maximize $L(X,p)$. Consider the economy instance with supply equal to $\sum_i x_{ij}$ for good j. Note that the saddle point conditions (11) are then satisfied with p and X for this instance and therefore they form an optimal solution to the corresponding EG-type convex program. Since any optimal solution to the convex program must also be an equilibrium, it follows that X must be a demand at price p as desired.

In fact it is easy to see that the converse of Lemma 3.1 is also true, that if for all p the demand set is equal to $\arg\max_{X\geq 0} L(X,p)$ then the economy is an EG economy. The saddle point conditions (11) are then exactly the same as the equilibrium conditions. In particular, the second condition holds if and only if for all j, $\sum_i x^*_{ij} \leq 1$, and whenever $p_j > 0$ this relation is an equality.

Lemma 3.2. $\nabla\phi(p) = \mathbb{1} - x(p) = -z(p)$.

PROOF. It is well known that if a convex function is defined as the maximum of many linear functions then the gradient is given by the gradient of the linear function providing this maximum. ϕ is indeed defined in this way and by Lemma 3.1 the arg max'es are given by the demands, or in other words the maximizing linear function $L(X,p)$ is the one defined using the demands. Thus $\nabla\phi(p) = \mathbb{1} - x(p) = -z(p)$.

The following convenient form for $\phi(p)$ was shown in [27], and will be used in the analyses of Fisher economies with Leontief and CES utilities.

Lemma 3.3. *For EG economies with linear, CES or Leontief utilities (and others) the dual objective can be written as*

$$\phi(p) = \sum_j p_j - \sum_i e_i \log(\nu_i) + \text{ a constant independent of } p,$$

where ν_i is the ratio of e_i to the optimal utility of i at price p, i.e., the minimum cost for obtaining one unit of utility.

PROOF. Recall that

$$\phi(p) := \max_{X\geq 0} L(X,p) = \max_{X\geq 0} \left\{ \sum_j p_j + \sum_i e_i \log u_i(x_i) - \sum_{i,j} p_j x_{ij} \right\},$$

where x_i denotes the demands of buyer i. From Lemma 3.1, for each i, an x_{ij} in the arg max above is buyer i's demand for good j and therefore $\sum_j p_j x_{ij}$ must be equal to e_i. Hence $\sum_{i,j} p_j x_{ij} = \sum_i e_i$ is a constant.

We also rewrite $\sum_i e_i \log u_i(x_i) = \sum_i -e_i \log[e_i/u_i(x_i)] + \sum_i e_i \log e_i$; then setting $\nu_i = e_i/u_i(x_i)$ gives ϕ in the desired form.

The following figure outlines and contrasts the key elements of the analyses of Fisher Economies with Leontief and complementary CES utilities.

Figure 2: The Two Main Subclasses of CPF Economies Being Analyzed: Fisher Economies with Leontief and with Complementary CES Utilities

Key Steps in the Analyses

1. Both are EG Economies: They each have a convex program characterizing its equilibrium.
2. Lemma 3.3: They each have a specific potential function for which tatonnement $\equiv$ gradient descent.

Leontief Utilities	*Complementary CES Utilities*
3. Use known convergence result on proximal gradient descent to show $O(1/t)$ convergence rate. (Lemma 4.1 + Theorem 2.2 $\Rightarrow$ Theorem 4.1)	3. Progress Bound: a lower bound on the reduction of the potential function value due to one round of tatonnement updates. (Lemma 5.1) 4. Distance Bound: an upper bound on the distance between current prices and the equilibrium prices via the potential function. (Lemma 5.2) 5. Combining the above two bounds to show a linear rate of convergence. (Theorem 5.1)

4. Fisher Economies with Leontief Utilities

In this section we consider Fisher economies in which every buyer has a Leontief utility. By Lemma 3.1, these economies are EG economies and hence CPF economies. By Lemma 3.3, the potential function can be written as $\phi(p) = \sum_j p_j - \sum_i e_i \log \nu_i$, where ν_i is the minimum cost buyer i has to pay to obtain one unit of utility. By (2), the maximum utility obtainable by buyer i equals $e_i \Big/ \sum_j b_{ij} p_j$. This utility is obtained by spending e_i money; consequently, the minimum cost for one unit of utility is $\sum_j b_{ij} p_j$. Thus the potential function is given by

$$\phi(p) = \sum_j p_j - \sum_i e_i \log \sum_j b_{ij} p_j.$$

We analyze update rule (7) with $d_h = 6 \cdot \gamma \cdot d_{\text{KL}}$ where d_{KL} is the KL-divergence, and γ is a market dependent parameter. This update rule, which minimizes $\nabla\phi(p^t) \cdot (p - p^t) + \gamma d_h(p, p^t)$, or equivalently minimizes $-z(p^t) \cdot (p - p^t) + \gamma\left(p \log p - p - (p - p^t) \log p^t\right)$, amounts to

$$p_j^{t+1} = p_j^t \cdot \exp\left(\frac{z_j^t}{\gamma}\right). \tag{12}$$

Our main result is to show an $O(1/\epsilon)$ convergence rate as specified in Theorem 4.1 below.

Notation. Let x^t denote the demands following the price updates at time t, and x° denote the initial demands. Let $\Delta p_j^t = p_j^{t+1} - p_j^t$ for all j.

In the rest of this section, we drop the superscript t when the meaning is clear from the context.

Theorem 4.1. *For a Leontief Fisher economy, for the sequence of prices defined by the update rule* (12) *with* $\gamma = 5 \cdot \max_j \left\{ x_j^\circ + 2 \cdot \sum_i \max_{k: b_{ik} > 0} \frac{b_{ij}}{b_{ik}} \right\}$, *for all* t,

$$\phi(p^t) - \phi(p^*) \le \frac{6\gamma d_{\scriptscriptstyle KL}(p^*, p^0)}{t}.$$

Proof. The result follows by applying Theorem 2.2. To do this, it suffices to ensure that Equation (10) holds for every price update (recall that $d_h = 6 \cdot \gamma \cdot d_{\text{KL}}$ here). By Lemma 4.1 below, it suffices to ensure $|\Delta p_j^t| \le p_j^t / 4$ for every price update. To this end, we require that $\gamma \ge 5 \cdot \max_{j,t}\{1, x_j^t\}$, where we are maximizing the x_j^t over all the time steps of the algorithm, for then $p_j^t e^{-1/5} \le p_j^{t+1} \le p_j^t e^{1/5}$ and $\left|\Delta p_j^t\right| / p_j^t \le e^{1/5} - 1 \le \frac{1}{4}$. Lemma 4.2 below shows that setting $\gamma = 5 \cdot \max_j \left\{ x_j^\circ + 2 \cdot \sum_i \max_{k: b_{ik} > 0} \frac{b_{ij}}{b_{ik}} \right\}$ suffices.

Comment. If a better bound on $\max_{j,t}\{1, x_j^t\}$ were known, that could be used to reduce the value of γ.

Complementing this upper bound, we show that in general the convergence rate is $\Omega(1/\sqrt{\epsilon})$ as specified in the next theorem (the proof is in Appendix A.1).

Theorem 4.2. *There is a 2-good, 2-buyer Leontief Fisher economy such that*

$$\phi(p^t) - \phi(p^*) = \Omega\left[\frac{\phi(p^0) - \phi(p^*)}{t^2}\right].$$

4.1. Proofs of Lemmas 4.1 and 4.2

Lemma 4.1 states that the bound in Equation (10) holds so long as the price update is not too large. The bound states that

$$\phi(p^{t+1}) - \ell_\phi(p^{t+1}; p^t) = \phi(p^{t+1}) - \phi(p^t) - \nabla\phi(p^t)(p^{t+1} - p^t) \le d_{\text{KL}}(p^{t+1}, p^t).$$

We can bound the expression on the LHS of the inequality by means of a power series expansion around p^t, yielding a bound consisting of second order terms which take the form

$$\frac{4}{3e_i}\sum_{k,\ell} x_{ik}x_{i\ell}|\Delta p_k| \cdot |\Delta p_\ell|.$$

It is convenient to bound this expression in turn using terms of the form $\frac{1}{p_j}(\Delta p_j)^2$ using Claim 4.1 below. Finally, it suffices to bound the terms $\frac{1}{p_j}(\Delta p_j)^2$ by a multiple of $d_{\text{KL}}(p_j + \Delta p_j, p_j)$, which we do using Claim 4.2.

Claims 4.1 and 4.2 are proved in Appendix A. In these claims, the index t on the prices and demands is implicit.

Claim 4.1.

$$\frac{1}{e_i}\sum_{j,k} x_{ik}x_{i\ell}|\Delta p_k| \cdot |\Delta p_\ell| \;\le\; \sum_j \frac{x_{ij}}{p_j}(\Delta p_j)^2.$$

Claim 4.2. *Suppose that* $|\Delta p_j| \le p_j/4$. *Then*

$$\frac{(\Delta p_j)^2}{p_j} \le \frac{9}{2} d_{KL}(p_j + \Delta p_j, p_j).$$

Lemma 4.1. *If* $|\Delta p_j^t| \le p_j^t/4$, *then Equation* (10) *holds with* $d_h = 6 \cdot \gamma \cdot d_{KL}$.

Proof. We will use the inequalities $x(1+x)^{-1} \ge x - \frac{4}{3}x^2$ for $|x| \le \frac{1}{4}$ and $\log(1+y) \le y$ for $|y| \le 1$, along with Claims 4.1 and 4.2.

$$\begin{aligned}
\phi(p^{t+1}) - \ell_\phi(p^{t+1}; p^t) &= \phi(p^{t+1}) - \phi(p^t) - \nabla\phi(p^t) \cdot (p^{t+1} - p^t) \\
&= \sum_j (p_j + \Delta p_j) - \sum_i e_i \log \sum_k b_{ik}(p_k + \Delta p_k) \\
&\qquad - \sum_j p_j + \sum_i e_i \log \sum_k b_{ik}p_k + \sum_j z_j \Delta p_j \\
&= \sum_j x_j \Delta p_j + \sum_i e_i \log \frac{\sum_k b_{ik}p_k}{\sum_k b_{ik}(p_k + \Delta p_k)} \\
&= \sum_j x_j \Delta p_j + \sum_i e_i \log\left[1 - \frac{\sum_k b_{ik}\Delta p_k}{\sum_k b_{ik}p_k}\left(1 + \frac{\sum_k b_{ik}\Delta p_k}{\sum_k b_{ik}p_k}\right)^{-1}\right].
\end{aligned}$$

Note that $\left|\frac{\sum_k b_{ik}\Delta p_k}{\sum_k b_{ik}p_k}\right| \le \frac{1}{4}$, as every $|\Delta p_k| \le \frac{1}{4}p_k$ by assumption. Thus, we can apply the bound $x(1+x)^{-1} \ge x - \frac{4}{3}x^2$ to $x = \frac{\sum_k b_{ik}\Delta p_k}{\sum_k b_{ik}p_k}$, yielding:

$$\phi(p^{t+1}) - \ell_\phi(p^{t+1};p^t) \;\le\; \sum_j x_j \Delta p_j + \sum_i e_i \log\left[1 - \frac{\sum_k b_{ik}\Delta p_k}{\sum_k b_{ik}p_k} + \frac{4}{3}\frac{\left(\sum_k b_{ik}\Delta p_k\right)\left(\sum_\ell b_{i\ell}\Delta p_\ell\right)}{\left(\sum_k b_{ik}p_k\right)\left(\sum_\ell b_{i\ell}p_\ell\right)}\right].$$

Now we use the bound $\log(1+y) \le y$, which applies as the second and third terms in the log are bounded by $\frac{1}{4}$ and $\frac{1}{12}$ respectively.

$$\begin{aligned}
\phi(p^{t+1}) - \ell_\phi(p^{t+1};p^t) \;&\le\; \sum_j x_j \Delta p_j + \sum_i e_i \left(-\frac{\sum_k b_{ik}\Delta p_k}{\sum_k b_{ik}p_k} + \frac{4}{3}\frac{\left(\sum_k b_{ik}\Delta p_k\right)\left(\sum_\ell b_{i\ell}\Delta p_\ell\right)}{\left(\sum_k b_{ik}p_k\right)\left(\sum_\ell b_{i\ell}p_\ell\right)}\right) \\
&\le\; \sum_j x_j \Delta p_j - \sum_k x_k \Delta p_k + \frac{4}{3}\sum_i \frac{1}{e_i}\sum_k x_{ik}\Delta p_k \sum_\ell x_{i\ell}\Delta p_\ell \qquad \text{(by (1))} \\
&\le\; \frac{4}{3}\sum_{i,j}\frac{x_{ij}}{p_j}(\Delta p_j)^2 \qquad \text{(by Claim 4.1)} \\
&=\; \frac{4}{3}\sum_j \frac{x_j}{p_j}(\Delta p_j)^2 \\
&\le\; 6\sum_j x_j \cdot d_{\text{KL}}(p_j + \Delta p_j, p_j) \qquad \text{(by Claim 4.2).}
\end{aligned}$$

Our global bound on x_{ij}^t follows from the fact that if x_{ij}^t is large enough, then the prices of the goods which buyer i desires will all increase, forcing x_{ij}^t to decrease. Lemma 4.2 specifies "large enough" precisely.

Lemma 4.2. *For all goods j and all times t, $x_j^t \le x_j^\circ + 2 \cdot \sum_i \max_{k:b_{ik}>0} \frac{b_{ij}}{b_{ik}}$. Hence*

$$\max_{j,t}\{1, x_j^t\} \le \max_j \left\{x_j^\circ + 2 \cdot \sum_i \max_{k:b_{ik}>0} \frac{b_{ij}}{b_{ik}}\right\}.$$

PROOF. If $x_{ij}^t \ge \max_{k:b_{ik}>0}\frac{b_{ij}}{b_{ik}}$, then for any k with $b_{ik} > 0$, $x_{ik}^t = \frac{b_{ik}}{b_{ij}}x_{ij}^t \ge 1$ and hence $x_k^t \ge x_{ik}^t \ge 1$. Thus, all these p_k increase, so x_{ij} must decrease, i.e. $x_{ij}^{t+1} \le x_{ij}^t$.

If $x_{ij}^t < \max_{k:b_{ik}>0}\frac{b_{ij}}{b_{ik}}$, then $x_{ij}^{t+1} \le 2x_{ij}^t < 2 \cdot \max_{k:b_{ik}>0}\frac{b_{ij}}{b_{ik}}$.

Given the two observations above, it is easy to show by induction that for all t,

$$x_{ij}^t \le \max\left\{x_{ij}^\circ, 2 \cdot \max_{k:b_{ik}>0}\frac{b_{ij}}{b_{ik}}\right\},$$

and hence

$$x_j^t \le \sum_i \left(x_{ij}^\circ + 2 \cdot \max_{k:b_{ik}>0}\frac{b_{ij}}{b_{ik}}\right) = x_j^\circ + 2 \cdot \sum_i \max_{k:b_{ik}>0}\frac{b_{ij}}{b_{ik}}.$$

5. Fisher Economies with Complementary CES Utilities

In this section we consider Fisher economies in which every buyer has a complementary CES utility. Again, these are EG and hence CPF economies. Broadly speaking, the analysis has the same structure as in the previous section. We prove a progress lemma, Lemma 5.1, analogous to Lemma 4.1; however, instead of relying on Theorem 2.2 to provide a convergence result, in Lemma 5.2 we prove an upper bound on the distance to the equilibrium point. Together, they demonstrate the linear convergence rate.

Suppose buyer i has utility function

$$u_i = (a_{i1}x_1^{\rho_i} + a_{i2}x_2^{\rho_i} + \cdots + a_{im}x_m^{\rho_i})^{1/\rho_i},$$

with $-\infty < \rho_i < 0$. Let $c_i := \rho_i/(\rho_i - 1)$. Recall that e_i denotes buyer i's budget. Let $b_{ij} := a_{ij}^{1-c_i}$ and $S_i := \sum_\ell b_{i\ell}p_\ell^{c_i}$. As is well known, the demand of buyer i for good j is given by

$$x_{ij} = e_i b_{ij} p_j^{c_i - 1} S_i^{-1}. \tag{13}$$

Substituting in (3) shows that the utility obtained by this demand equals $e_i S_i^{-1/c_i}$. It follows that the minimum cost for one unit of utility is S_i^{1/c_i}. Thus, by Lemma 3.3, the potential function for this economy ϕ is

$$\phi(p) = \sum_j p_j - \sum_i e_i \log S_i^{1/c_i}.$$

For these economies, we analyze the update rule

$$p_j^{t+1} = p_j^t \cdot \exp\left(\frac{z_j^t}{\gamma_j^t}\right). \tag{14}$$

Note that the weights γ_j^t are allowed to change from one time step to the next; in particular, we use the weight $\gamma_j^t = 5 \cdot \max\{1, x_j^t\}$.[11] This seems a very natural distributed rule, and indeed a linearization of this rule, $p_j^{t+1} = p_j^t[1 + \lambda \cdot \min\{1, z_j\}]$,[12] was used in earlier papers by Cole and Fleischer [24] and Cheung et al. [18].

Notation. Let $c := \max_i c_i$. Let Δp_j denote $p_j^{t+1} - p_j^t$. Henceforth, the superscript t on all the parameters except prices will be implicit.

Our main result for complementary CES Fisher economies is stated in Theorem 5.1 below. We show that $\phi(p^t) - \phi(p^*)$ reduces by at least a $1-\mu$ factor at each time step, where $0 < \mu < 1$ depends on the following parameters of the economy: c, the total money M, the initial prices p^0, and the equilibrium prices p^*. This follows by showing that the potential function in this case satisfies a new progress property, as specified in Lemma 5.1, and a new upper bound on the distance to the equilibrium, as specified in Lemma 5.2. These lemmas are stated below and proved in the next two subsections. To obtain the new bounds we introduce a new polynomial function which will be used when bounding the log term. This function, $h_c(\cdot)$, and a relevant parameter r are specified next.

For all j, define $r_j^t := p_j^*/p_j^t$ and $\bar{r}_j = \sup_t\{r_j^t\}$. By Lemma 5.5, which will be stated in Section 5.3, $\bar{r}_j$ is finite. Also define $h_c(r)$ to be the following function of r: $h_c(r) = \frac{1-r^c+c(r-1)}{(r-1)^2}$ for any $r \geq 0$ except $r = 1$, and $h_c(1) = c(1-c)/2$. Finally, define $H_c(r) := h_c(r)/c$. Some basic properties of $h_c(r)$ are shown in Claim 5.1 below.

Lemma 5.1.

$$\phi(p^t) - \phi(p^{t+1}) \geq \frac{1}{2}\sum_j \frac{z_j^2 p_j^t}{\gamma_j^t}.$$

Lemma 5.2.

$$\phi(p^t) - \phi(p^*) \leq \max_j \left\{10, \frac{5}{2H_c(\bar{r}_j)}\right\} \cdot \sum_j \frac{z_j^2 p_j^t}{\gamma_j^t}.$$

[11] Any greater value for γ_j would work too, but would entail a proportionate change to the bound in Lemma 5.2.

[12] The λ replaces the constant of 5 used here, as a greater range of values for this parameter is needed in economies of substitutes.

Claim 5.1. *Suppose $0 < c < 1$. Then*

i. For fixed $0 < c < 1$, $h_c(r)$ is continuous at $r = 1$. $h_c(r) > 0$ for all $r \geq 0$, and $h_c(r)$ is a decreasing function of r for $r \geq 0$.

ii. For fixed $r \geq 0$, $h_c(r)/c$ is a decreasing function of c for $c \in (0, 1)$.

Proof. By simple calculus.

Theorem 5.1. *For a complementary CES Fisher economy, for the sequence of prices p^t defined by the update rule (14) with $\gamma_j^t = 5 \cdot \max\{1, x_j^t\}$, for all t,*

$$\phi(p^t) - \phi(p^*) \leq (1 - \Theta(1))^t \left[\phi(p^t) - \phi(p^0)\right].$$

In other words, for any $\epsilon > 0$, $\phi(p^t) - \phi(p^*) \leq \epsilon[\phi(p^t) - \phi(p^0)]$, if $t = \Omega(\log(1/\epsilon))$.

Proof. (of Theorem 5.1) We show that $\phi(p^t) - \phi(p^*)$ drops by a constant factor in every step as follows:

$$\begin{aligned}
\phi(p^{t+1}) - \phi(p^*) &= \phi(p^t) - \phi(p^*) - [\phi(p^t) - \phi(p^{t+1})] \\
&\leq \phi(p^t) - \phi(p^*) - \frac{1}{2}\sum_j \frac{z_j^2 p_j^t}{\gamma_j^t} \quad \text{(by Lemma 5.1)} \\
&\leq [\phi(p^t) - \phi(p^*)] \left[1 - \frac{1}{2}\left(\max_j \left\{10, \frac{5}{2H_c(\bar{r}_j)}\right\}\right)^{-1}\right] \quad \text{(by Lemma 5.2)} \\
&= [\phi(p^t) - \phi(p^*)] \left[1 - \min_j \left\{\frac{1}{20}, \frac{h_c(\bar{r}_j)}{5c}\right\}\right].
\end{aligned}$$

By Claim 5.1(i), $\min_j \left\{\frac{1}{20}, \frac{h_c(\bar{r}_j)}{5c}\right\}$ is strictly positive. Consequently,

$$\phi(p^{t+1}) - \phi(p^*) = (1 - \Theta(1))[\phi(p^t) - \phi(p^*)].$$

5.1. Proof of Lemma 5.1: Good Progress on a Price Update

We begin by showing that the price update is bounded by $\frac{1}{4}p_j^t$ (Claim 5.2). Modulo this condition, we bound $\phi(p + \Delta p) - \ell_\phi(p + \Delta p; p)$ (Lemma 5.3). Lemma 5.1 then follows fairly easily.

Claim 5.2. $|p_j^{t+1} - p_j^t| \leq \frac{1}{4}p_j^t$.

Proof. $|p_j^{t+1} - p_j^t| \leq (e^{1/5} - 1)p_j^t \leq \frac{1}{4}p_j^t$.

Lemma 5.3. *Suppose that for all j, $|\Delta p_j|/p_j \leq \frac{1}{4}$. Then*

$$\phi(p + \Delta p) - \ell_\phi(p + \Delta p; p) \leq 2\sum_j \frac{x_j}{p_j}(\Delta p_j)^2.$$

Proof. (of Lemma 5.1). Recall that $\Delta p_j = p_j^{t+1} - p_j^t$ and that $p_j^{t+1} = p_j^t \cdot \exp\left(\frac{z_j}{\gamma_j}\right)$. By Lemma 5.3,

$$\phi(p^t) - \phi(p^{t+1}) \;\geq\; \sum_j z_j \left(p_j^{t+1} - p_j^t\right) - 2\sum_j \frac{x_j}{p_j^t}\left(p_j^{t+1} - p_j^t\right)^2. \tag{15}$$

Next, using the formula for p^{t+1} and the fact that $\gamma_j \geq 5x_j$ gives the bound:

$$\begin{aligned}
\phi(p^t) - \phi(p^{t+1}) \;&\geq\; \sum_j z_j p_j^t \left(\exp(z_j/\gamma_j) - 1\right) - \frac{2}{5}\sum_j \gamma_j p_j^t \left(\exp(z_j/\gamma_j) - 1\right)^2 \\
&= \sum_j \gamma_j p_j^t \left[\frac{z_j}{\gamma_j}\left(\exp(z_j/\gamma_j) - 1\right) - \frac{2}{5}\left(\exp(z_j/\gamma_j) - 1\right)^2\right].
\end{aligned}$$

To finish, we use the following bound: if $|y| \leq 1/5$, then $y\left(\exp(y) - 1\right) - \frac{2}{5}\left(\exp(y) - 1\right)^2 \geq \frac{1}{2}y^2$. This yields the lemma, on setting $y = z_j/\gamma_j$.

PROOF. (of Lemma 5.3). We will use the following two bounds. First, a bound on $\log(1+\epsilon)$, namely:

$$\log(1+\epsilon) \geq \epsilon - \frac{2}{3}\epsilon^2, \text{ when } |\epsilon| \leq \frac{7}{24}. \tag{16}$$

And second, a bound on the following polynomial, which follows from a simple power series expansion: if $|\Delta p_j|/p_j \leq 1/4$ and $0 \leq c \leq 1$,

$$(p_j + \Delta p_j)^c \geq p_j^c + c p_j^{c-1}(\Delta p_j) - \frac{2}{3} c p_j^{c-2}(\Delta p_j)^2. \tag{17}$$

Let D_ϕ denote $\phi(p+\Delta p) - \ell_\phi(p+\Delta p; p)$, for short. Recall that $S_i(p) = \sum_\ell b_{i\ell} p_\ell^{c_i}$. Then:

$$\begin{aligned}
D_\phi &= \phi(p+\Delta p) - \phi(p) + \sum_j z_j \Delta p_j \\
&= \sum_j \Delta p_j + \sum_j z_j \Delta p_j - \sum_i \frac{e_i}{c_i} \log \frac{S_i(p+\Delta p)}{S_i(p)}. \\
&= \sum_j x_j \Delta p_j - \sum_i \frac{e_i}{c_i} \log \left(\frac{\sum_\ell b_{i\ell}(p_\ell + \Delta p_\ell)^{c_i}}{S_i(p)} \right).
\end{aligned}$$

As $\rho_i < 0$, $0 < c_i < 1$. So we can apply (17), yielding:

$$D_\phi \leq \sum_j x_j \Delta p_j - \sum_i \frac{e_i}{c_i} \log \left(1 + \frac{\sum_\ell b_{i\ell} c_i p_\ell^{c_i - 1}(\Delta p_\ell)}{S_i(p)} - \frac{\frac{2}{3}\sum_\ell b_{i\ell} c_i p_\ell^{c_i-2}(\Delta p_\ell)^2}{S_i(p)} \right).$$

Recalling from (13) that $x_{i\ell} = e_i b_{i\ell} p_\ell^{c_i - 1}/S_i(p)$, yields:

$$D_\phi \leq \sum_j x_j \Delta p_j - \sum_i \frac{e_i}{c_i} \log \left(1 + \sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta p_\ell) - \frac{2}{3}\sum_\ell c_i \frac{x_{i\ell}}{p_\ell e_i}(\Delta p_\ell)^2 \right).$$

On applying (16), which we can do as $\sum_\ell x_{i\ell} p_\ell \leq e_i$, $c_i \leq 1$, and $|\Delta p_\ell|/p_\ell \leq \frac{1}{4}$, we obtain the bound:

$$\begin{aligned}
D_\phi \leq\ & \sum_j x_j \Delta p_j - \sum_i \frac{e_i}{c_i} \left(\sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta p_\ell) - \frac{2}{3}\sum_\ell c_i \frac{x_{i\ell}}{p_\ell e_i}(\Delta p_\ell)^2 \right) \\
& + \sum_i \frac{e_i}{c_i}\frac{2}{3} \left(\sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta p_\ell) - \frac{2}{3}\sum_\ell c_i \frac{x_{i\ell}}{p_\ell e_i}(\Delta p_\ell)^2 \right)^2 \\
=\ & \frac{2}{3}\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{2}{3}\sum_i \frac{c_i}{e_i} \left(\sum_\ell x_{i\ell}(\Delta p_\ell)\left(1 - \frac{2\Delta p_\ell}{3 p_\ell}\right) \right)^2.
\end{aligned}$$

Now recall that $|\Delta p_\ell|/p_\ell \leq \frac{1}{4}$, to give the bound:

$$\begin{aligned}
D_\phi &\leq \frac{2}{3}\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{2}{3}\sum_i \frac{c_i}{e_i}\left(\sum_\ell x_{i\ell}|\Delta p_\ell| \cdot \frac{7}{6} \right)^2 \\
&= \frac{2}{3}\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{49}{54}\sum_i \frac{1}{e_i}\left(\sum_\ell x_{i\ell}|\Delta p_\ell| \right)^2 \quad \text{(as } c_i \leq 1\text{)} \\
&= \frac{2}{3}\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{49}{54}\sum_i \frac{1}{e_i}\sum_{j,k} x_{ij} x_{ik} |\Delta p_j||\Delta p_k| \\
&\leq \left(\frac{2}{3} + \frac{49}{54} \right) \sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 \quad \text{(by Claim 4.1)} \\
&\leq 2\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2.
\end{aligned}$$

5.2. Proof of Lemma 5.2: An Upper Bound on the Distance to Equilibrium

Our proof of Lemma 5.2 uses the following bound on $\phi(p^*) - \ell_\phi(p^*; p)$, stated in Lemma 5.4 below.

Lemma 5.4. *Suppose that $r_j^t = p_j^*/p_j^t \le \bar{r}_j$ for all j. Then*

$$\phi(p^*) - \ell_\phi(p^*; p) \ge \sum_j \frac{h_c(\bar{r}_j)}{c} x_j \frac{(p_j^* - p_j)^2}{p_j}.$$

Proof. (of Lemma 5.2). Note that $H_c(\bar{r}_j) = h_c(\bar{r}_j)/c$. By Lemma 5.4:

$$\begin{aligned}
\phi(p^t) - \phi(p^*) &= \ell_\phi(p^*, p^t) - \phi(p^*) - \nabla\phi(p^t) \cdot (p^* - p^t) \\
&\le \sum_j z_j(p_j^* - p_j^t) - \sum_j H_c(\bar{r}_j) x_j \frac{(p_j^* - p_j^t)^2}{p_j^t} \\
&\le \max_{p'} \sum_j \left(z_j(p'_j - p_j^t) - H_c(\bar{r}_j) x_j \frac{(p'_j - p_j^t)^2}{p_j^t} \right).
\end{aligned}$$

There are two cases.

<u>Case 1:</u> $0 \le x_j \le 1/2$.

Then $-1 \le z_j \le -1/2$ and hence $z_j \ge -2z_j^2$. Thus

$$z_j(p'_j - p_j^t) - H_c(\bar{r}_j) x_j \frac{(p'_j - p_j^t)^2}{p_j^t} \le -z_j p_j^t \le 2 z_j^2 p_j^t = 2\gamma_j \frac{z_j^2 p_j^t}{\gamma_j}.$$

As $x_j \le 1/2 < 1$, $2\gamma_j = 10$. Hence

$$z_j(p'_j - p_j^t) - H_c(\bar{r}_j) x_j \frac{(p'_j - p_j^t)^2}{p_j^t} \le 10 \frac{z_j^2 p_j^t}{\gamma_j}.$$

<u>Case 2:</u> $x_j \ge 1/2$.

$z_j(p'_j - p_j^t) - H_c(\bar{r}_j) x_j \frac{(p'_j - p_j^t)^2}{p_j^t}$ is a quadratic function of $(p'_j - p_j^t)$. The quadratic function is maximized when $(p'_j - p_j^t) = \frac{z_j p_j^t}{2H_c(\bar{r}_j) x_j}$, with its maximum value being $\frac{z_j^2 p_j^t}{4H_c(\bar{r}_j) x_j} = \frac{\gamma_j}{4H_c(\bar{r}_j) x_j} \frac{z_j^2 p_j^t}{\gamma_j}$.

As $x_j \ge 1/2$ and $\gamma_j = 5 \cdot \max\{1, x_j\}$, $\gamma_j / x_j \le 10$. Hence

$$z_j(p'_j - p_j^t) - H_c(\bar{r}_j) x_j \frac{(p'_j - p_j^t)^2}{p_j^t} \le \frac{5}{2H_c(\bar{r}_j)} \frac{z_j^2 p_j^t}{\gamma_j}.$$

Combining the two cases yields the lemma.

Proof. (of Lemma 5.4). By a direct calculation,

$$(p_j^*)^{c_i} = p_j^{c_i} + c_i p_j^{c_i - 1}(p_j^* - p_j) - h_{c_i}(r_j^t) \cdot (p_j^* - p_j)^2.$$

By Claim 5.1(i), $h_{c_i}(r_j^t) \ge h_{c_i}(\bar{r}_j)$. Thus

$$(p_j^*)^{c_i} \le p_j^{c_i} + c_i p_j^{c_i - 1}(p_j^* - p_j) - h_{c_i}(\bar{r}_j) \cdot (p_j^* - p_j)^2.$$

To avoid clutter, we omit the superscript t on the prices.

Let $\Delta^* p_j = p_j^* - p_j$. Then

$$\phi(p^*) - \ell_\phi(p^*; p) = \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i} \log \left(\frac{\sum_\ell b_{i\ell} (p_\ell^*)^{c_i}}{S_i(p)} \right).$$

Recalling that $S_i(p) = \sum_l b_{i\ell}(p_\ell)^{c_i}$ and using the upper bound on $(p_j^*)^{c_i}$ derived above, gives:

$$\begin{aligned}
\phi(p^*) - \ell_\phi(p^*;p) \;&\geq\; \sum_j x_j \Delta^* p_j \;-\; \sum_i \frac{e_i}{c_i} \log\left(1 + \frac{\sum_\ell b_{i\ell} c_i p_\ell^{c_i-1}(\Delta^* p_\ell)}{S_i(p)} - \frac{\sum_\ell b_{i\ell} h_{c_i}(\bar{r}_j) p_\ell^{c_i-2}(\Delta^* p_\ell)^2}{S_i(p)}\right) \\
&=\; \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i} \log\left(1 + \sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta^* p_\ell) - \sum_\ell h_{c_i}(\bar{r}_j) \frac{x_{i\ell}}{p_\ell e_i}(\Delta^* p_\ell)^2\right).
\end{aligned}$$

Note that the argument for the log is positive (as it is an upper bound for $S_i(p^*)/S_i(p)$). We apply the bound $\epsilon \geq \log(1+\epsilon)$ for $\epsilon \geq -1$ to give:

$$\begin{aligned}
\phi(p^*) - \ell_\phi(p^*;p) &\geq \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i}\left(\sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta^* p_\ell) - \sum_\ell h_{c_i}(\bar{r}_j)\frac{x_{i\ell}}{p_\ell e_i}(\Delta^* p_\ell)^2\right) \\
&= \sum_i \sum_\ell \frac{h_{c_i}(\bar{r}_j)}{c_i} x_{i\ell} \frac{(\Delta^* p_\ell)^2}{p_\ell} \\
&\geq \sum_i \sum_\ell \frac{h_c(\bar{r}_j)}{c} x_{i\ell} \frac{(\Delta^* p_\ell)^2}{p_\ell} \qquad \text{(by Claim 5.1(ii))} \\
&= \sum_j \frac{h_c(\bar{r}_j)}{c} x_j \frac{(p_j^* - p_j)^2}{p_j}.
\end{aligned}$$

5.3. Bounding r_j^t

Let $p_{\text{U}} = \max_j\{p_j^\circ\}$, the maximum initial price, $U = \max\{p_{\text{U}}, M\}$, and $L^* = \min_j\{p_j^*\}$. The following bound is proved in Appendix B.

Lemma 5.5. *Let $\overline{U} = 2U$. For all t,*

$$r_j^t = \frac{p_j^*}{p_j^t} \leq 2 \cdot \max\left\{\frac{p_j^*}{p_j^\circ}, \left(\frac{L^*}{\overline{U}}\right)^{\min_i \rho_i}\right\}.$$

6. Fisher Economies with Substitute CES Utilities

The analysis in Section 5 can be extended to Fisher economies with substitute CES utilities, i.e., CES utility functions with parameters $\rho \geq 0$. Cole and Fleischer [24] showed that tatonnement converges in these economies (and others) via a different potential function. For completeness, we reprove this result here with the technique developed in Section 5. We will prove lemmas similar to Lemmas 5.1, 5.2, 5.3, 5.4 and then prove a theorem similar to Theorem 5.1, which is Theorem 6.1 below. All proofs are deferred to Appendix C.

For substitute CES utilities, the parameter $c_i = \rho_i/(\rho_i - 1)$ is negative, while it is positive in the complementary case. Due to the sign switch, some of the proofs of the lemmas for this result differ from the corresponding proofs in Section 5.

We also need to change the parameter γ_j^t. Let $c_{\min} = \min c_i$. We set

$$\gamma_j^t = 5(1 + |c_{\min}|) \cdot \max\left\{1, x_j^t\right\}.$$

Note that as $\rho \nearrow 1$, $\gamma_j^t \nearrow +\infty$, i.e., the step size shrinks to zero, which is as one would expect for the utility functions at the limit as $\rho \nearrow 1$ are the linear utilities, and for these utilities a discrete tatonnement will not converge to the equilibrium in general, however small the step size.

Theorem 6.1. *For a substitute CES Fisher economy, for the sequence of prices p^t defined by update rule (14) with $\gamma_j^t = 5(1 + |c_{\min}|) \cdot \max\{1, x_j^t\}$, for all t,*

$$\phi(p^t) - \phi(p^*) \leq (1 - \Theta(1))^t \left[\phi(p^t) - \phi(p^0)\right].$$

In other words, for any $\epsilon > 0$, $\phi(p^t) - \phi(p^*) \leq \epsilon[\phi(p^t) - \phi(p^0)]$, if $t = \Omega(\log(1/\epsilon))$.

7. Convergence of Continuous Time Tatonnement

We begin by explaining how to formulate a continuous version of the tatonnment based on the Bregman divergence. We follow this with an overview of our analysis, which we then expand upon.

Continuous Time Tatonnement via Differential Inclusion. A continuous version of tatonnement is a trajectory in the price space which, to be notationally consistent with the discrete version, is denoted by p^t for all $t \in \mathbb{R}_+$. Classically, the trajectory is defined by specifying a differential equation $\frac{dp}{dt} = F(t, p(t))$ for all t, which we also call the "update rule". We define a family of update rules derived from gradient descent. As before, let h be a strictly convex differentiable function. The natural way to specify the differential equation is

$$\begin{aligned} p(\epsilon) &:= \arg\min_p \left\{\nabla\phi(p^t)\cdot(p-p^t) + \tfrac{1}{\epsilon}d_h(p;p^t)\right\} \\ \frac{dp_j}{dt} &:= \lim_{\epsilon\to 0}\frac{p_j(\epsilon)-p_j^t}{\epsilon}. \end{aligned}$$

However, in the economies we consider, the demand function of an agent can be multi-valued at a price vector[13], and hence $\nabla\phi(p^t)$ can also be a set of multiple elements, namely the set of subgradients of ϕ at p^t. Since $\nabla\phi(p^t)$ can be multi-valued, $p(\epsilon)$ and hence $\frac{dp_j}{dt}$ can be too. To resolve this, as is standard, we employ *differential inclusions*, which are a generalization of differential equations. In brief, a differential inclusion is a system which allows $\frac{dp}{dt}$ to take any value from a set. We specify our class of differential inclusions *in the domain* $\mathbb{R}_+^m$, as follows:

$$p^t(\vec{v},\epsilon) := \arg\min_p \left\{\vec{v}\cdot(p-p^t) + \tfrac{1}{\epsilon}d_h(p;p^t)\right\} \tag{18}$$

$$F(p^t) := \left\{\lim_{\epsilon\to 0}\frac{p^t(\vec{v},\epsilon)-p^t}{\epsilon}\ \middle|\ \vec{v}\in\nabla\phi(p^t)\right\} \tag{19}$$

$$\frac{dp}{dt} :\in F\left(p^t\right). \tag{20}$$

Overview of the Analysis. Our goal is to show that a general class of continuous tatonnements, specified by (18)–(20), converges toward equilibrium prices for a broad subclass of CPF economies. To do this we need to identify conditions that ensure the *global existence* of a trajectory — in other words, we need to show that a solution to (20) exists for all $t \in [0, +\infty)$. We will motivate and provide conditions on the function h and the economies which guarantee global existence. Having obtained such a trajectory, we then prove that, modulo these conditions, it must converge to an equilibrium point.

The first set of conditions (Definition 7.1), which we call *allowability*, concern h alone. The main reason for its introduction is to exclude those update rules which can cause a price to soar to infinity within a finite time, which is possible for some unnatural choices of h; see Example D.2 in the appendix for a concrete example.

The second set of conditions (Definition 7.2), which we call *controllability*, concern ϕ and the tatonnement rule. They will be used to demonstrate the global existence of a solution to the system (18)–(20). Informally, controllability excludes scenarios in which some absolute excess demand (normalized by $h''(p_j)$) or price can blow up to infinity too quickly. If these bad scenarios do not occur, we can ensure that local existence solutions, as guaranteed by a standard theorem about differential inclusions, can be combined indefinitely over time to form a global solution.

Theorem 7.1. *Let $\phi : \mathbb{R}_+^m \to \mathbb{R}$ and $p^t \in \mathbb{R}_+^m$ be defined by* (18)–(20)*. Suppose that ϕ is convex, h is allowable and that ϕ together with the rule given by* (18)–(20) *is controlled. Then, for any starting bounded*

[13] An example: if a buyer has utility function $u(x_1,x_2) = x_1 + 3x_2$ and budget 40, then at prices $(p_1,p_2) = (2,6)$, the buyer optimizes her utility by purchasing $(x_1,x_2) = (20-3y, y)$, for any $y \in [0, 20/3]$.

demand price vector p^0 such that for all j, $h''(p_j^0)$ is finite, if the economy is a Fisher economy, then $\lim_{t\to\infty} p^t = p^*$, *where p^* is a minimizer of ϕ.*

In Arrow-Debreu economies, if in addition d_h is the KL-divergence then

$$\lim_{t\to\infty} \hat{p}^t = \hat{p}^*,$$

where $\hat{p}$ is the normalized price vector of p, i.e., for any price vector p with at least one positive price, the corresponding normalized price vector $\hat{p}$ is given by $\hat{p}_j = p_j / \left(\sum_\ell p_\ell\right)$.

For any CPF economy, by definition, there exists a ϕ such that $-\nabla\phi(p) = z(p)$. Substituting z for $-\nabla\phi$ in (18)–(20) gives a tatonnement update rule for which, by Theorem 7.1, the prices converge to an equilibrium.

Roadmap. In Section 7.1 we give the definitions of the necessary conditions: allowability and controllability. Next, in Section 7.2.1, we prove the existence of a *local* solution starting at the trajectory origin; this uses allowability alone. Then, in Section 7.2.2, we prove global existence. Finally, in Section 7.3, we demonstrate convergence to an equilibrium. We will state only the most important intermediate steps, and defer many of the technical proofs to Appendix D. For the reader's convenience, we also state some standard definitions related to differential inclusions in the appendix.

7.1. The necessary conditions

We will limit the study to the special case where h is a *separable* function, i.e., it is of the form $\sum_j h(p_j)$, for a 1-dimensional function $h : \mathbb{R} \to \mathbb{R}$. We will also need h to be twice differentiable. It may be that $h'(0) = -\infty$, but by the convexity of h, this is the only argument for which h' might be infinite. And if $h'(0) = -\infty$ then $h''(0) = \infty$.

As we will see in Lemma 7.2, $\frac{dp_j}{dt} = -\nabla_j\phi(p^t)/h''(p_j^t)$ if $\nabla_j\phi(p^t)$ and $h''(p_j^t)$ are finite. In order to make progress, we will need that $h''(p_j^t) \neq \infty$. (We want to stop because $\nabla_j\phi(p^t) = 0$ but not because $h''(p_j^t)$ is ∞.) In addition, we need any solution trajectory to stay away from prices at which $\nabla_j\phi(p^t)$ is infinite ($h''(p_j^t) = 0$ is ruled out as h is strictly convex). Suitable constraints on h, called *allowable* h, suffice to achieve these objectives.

7.1.1. Allowable h

As we will show in Lemma 7.1, allowability ensures that the prices and h'' remain finite at all times. Assumptions B1 and B2 in the definition of allowability ensure the property for h'' and Assumptions A1 and A2 together with B2 achieve it for prices.

Note that we want to ensure that h'' remains finite, for if $h''(p_j) = \infty$ with $\nabla_j\phi(p^t)$ finite, then by Lemma 7.2, $\frac{dp_j^t}{dt} = 0$ which means p_j remains unchanged, which is undesirable if $\nabla_j\phi(p^t) \neq 0$. While if both h'' and $\nabla\phi(p^t)$ are infinite, then $\frac{dp_j^t}{dt}$ is not well defined.

Definition 7.1. *$h(p)$ is* allowable *if h is twice differentiable and strictly convex (hence $h''(p) > 0$), $h''(p)$ is finite if $p > 0$, $1/h''$ is continuous, and either*

A1. The economy is a Fisher economy, or
A2. $\int_p^\infty h''(q)dq = \infty$ for all $p > 0$,

and in addition either

B1. $h''(p)$ is finite for all p, or
B2. $\int_0^p h''(q)dq = \infty$ for all $p > 0$; in this case, we say h is controlling.

Henceforth, we assume that h is allowable.

We note that two of the most commonly used Bregman divergences satisfy the above assumptions. The first one uses $h(p_j) = \frac{1}{2}p_j^2$; thus $h''(p_j) = 1$; hence $\frac{dp_j}{dt} = -\nabla_j\phi(p)$. Also, for $p > 0$, $\int_p^\infty h''(q)dq = \infty$, so conditions A2 and B1 are satisfied. The second one, which is the KL-divergence, uses $h(p_j) = p_j \log p_j - p_j$, $h'(p_j) = \log p_j$ and $h''(p_j) = 1/p_j$. Hence $\frac{dp_j}{dt} = -p_j\nabla_j\phi(p)$. Also, $\int_0^p \frac{dq}{q} = \log p - \log 0 = \infty$ and for $p > 0$, $\int_p^\infty \frac{dq}{q} = \log\infty - \log p = \infty$, so conditions A2 and B2 are satisfied.

Lemma 7.1.

i. *Suppose that $h''(p_j^0)$ is finite. If h is allowable then $h''(p_j^t)$ is finite for all $t \geq 0$.*
ii. *Suppose that p^0 is finite. If h is allowable then p^t is finite for all $t \geq 0$.*

Example D.2 in the appendix shows that a price may blow up to $+\infty$ in finite time if condition A2 is violated.

7.1.2. Controllability

To understand the definition of controllability, we first need to characterize the set of optimal bundles of an agent at price vector p. There are two possibilities:

1. Every bundle includes at least one good having infinite demand. Then we say that p is an *unbounded demand* price vector. Note that this good must have price zero, and by Lemma 7.1(i) this price can be reached only if $h''(0)$ is finite. The controllability requirement ensures that in this case the tatonnement trajectory does not reach any unbounded demand price vector. (If $h''(0)$ is infinite this is already ensured by Lemma 7.1(i)).
2. All the demands in at least one bundle are finite. Then we say that p is a *bounded demand* price vector. Note that if p includes a zero price, $p_j = 0$ say, then an optimal bundle can have an infinite demand for good j; but p is a bounded demand price vector if for all such j, the demand for good j could also be finite.
   For instance, in a Leontief Fisher economy, an equilibrium price vector may include a zero price but it will be a bounded demand price vector; clearly, we want the tatonnement trajectory to be able to converge to it. Furthermore, in this case, as the tatonnement proceeds, we want the agent's sequence of optimal bundles to always have bounded demands, and further these bounds should apply throughout the tatonnement process.

We are now ready to define controllability.

Definition 7.2. *Let ϕ be a potential function and $\mathcal{T}$ a continuous tatonnement rule. The pair $(\phi, \mathcal{T})$ is* controlled*, if for any bounded demand starting price vector p^0 and any finite time $\bar{t} \geq 0$, there are finite bounds $b(p^0, \bar{t})$ and $c(p^0, \bar{t})$ such that for any tatonnement trajectory from time 0 to time $\bar{t}$ induced by* (18)–(20)*, there exists a neighbourhood Ω of the trajectory in which for any $p \in \Omega$ and for any j,*

1. *$|\nabla_j \phi(p)/h''(p_j)| \leq b(p^0, \bar{t})$ and $p \leq c(p^0, \bar{t})$;*
2. *$\lim_{t \nearrow \bar{t}} b(p^0, t)$ and $\lim_{t \nearrow \bar{t}} c(p^0, t)$ are finite*[14]*,*

i.e. both the prices and the rate of change of the prices remain bounded throughout the tatonnement process up to and including time $\bar{t}$.

In Claim D.1 in the appendix, we will show that controllability is obeyed by Fisher economies with CES, Leontief and linear utilities along with *any* tatonnement rule (i.e., even if h is not controlling); this result was extended to Fisher economies with nested CES utilities (see [51] for a definition) in Cheung's thesis [15]. Also, in Claim D.2, we will show that if h is controlling (recall Definition 7.1) then $(\phi, \mathcal{T})$ is controlled. However, it is not clear whether the latter result applies to all economies or even to all EG economies.

7.2. Existence of a Solution for (20)

With h being separable, the minimization in (18) becomes an independent minimization problem for each good j. We let $d_h(p_j, q_j)$ denote $h(p_j) - h(q_j) - h'(q_j)(p_j - q_j)$, the one dimensional version of the Bregman divergence. Note that as h is convex,

$$d_h(p_j, q_j) \geq 0, \tag{21}$$

and, by the strict convexity of h,

$$\text{if } p_j \neq q_j, \quad d_h(p_j, q_j) > 0. \tag{22}$$

[14] Without loss of generality, we may assume that $b(p^0, t), c(p^0, t)$ are increasing functions of t, so the limits exist.

Lemma 7.2. *For all j, if $\nabla_j \phi(p^t)$ and $h''(p^t_j)$ are finite, then*

$$\frac{dp^t_j}{dt} = \frac{-\nabla_j \phi(p^t)}{h''(p^t_j)}.$$

Theorem 7.2, a standard theorem on differential inclusions, guarantees the existence of a solution over a time interval $[0, T]$, for some $T > 0$.

Let $B(p_0, \rho)$ denote the closed ball around p_0 with radius ρ.

Theorem 7.2 ([62, p. 96–103]). *Let $\frac{dp}{dt} \in F(p(t))$ be a differential inclusion, where $F : P \to \mathbb{P}(\mathbb{R})$ is upper semi-continuous at every $p' \in B(p_0, \rho)$ for some $\rho > 0$. Suppose that $F(p')$ is convex and compact for every $p' \in B(p_0, \rho)$, and there exists a finite κ such that $\sup_{\vec{z} \in F(p')} ||z|| \le \kappa$ for every $p' \in B(p_0, \rho)$. Then for $0 \le t \le \rho/\kappa$, there exists an absolutely continuous solution $p(t)$ to the differential inclusion with $p(0) = p_0$.*

To apply this theorem we would need that all possible demands in a neighborhood $\mathcal{N}$ of the tatonnement trajectory be bounded. But this need not be the case, because the equilibrium and hence the desired end of the trajectory may have some prices at zero. We will sidestep this difficulty by only considering demands that lie within some bound $b > 0$; in order for the differential inclusion to have a solution we need to ensure that the resulting demand sets for $p \in \mathcal{N}$, denoted $F_b(p)$, are all non-empty. This is ensured by the controllability assumption.

Furthermore, Theorem 7.2 yields a solution only up to some time $\bar{t} = \rho/\kappa$, which we call a *local* solution, but we need a *global* solution, a solution for all $t \in [0, \infty)$. Controllability will allow the local solution to be repeatedly extended so as to provide the desired global solution.

7.2.1. Local Existence of a Solution to (20)

We start by showing that there is a solution to (20) for some time interval $[0, \bar{t}]$, albeit modulo some additional assumptions. Then we will show how to extend the solution to arbitrarily large t and remove these assumptions.

In order to apply Theorem 7.2 to (20), we need its right hand side ($-\nabla_j \phi(p)/h''(p_j)$ when $\nabla_j \phi(p)$ is finite) to be convex, compact and upper semi-continuous in any ball $B(p_0, \rho)$ we consider. The difficulty we face is that when some prices are zero, the corresponding demands can be infinite, and then compactness will not hold for such price vectors.

To restore compactness we modify F by intersecting it with a ball of radius $b > 0$: let $F_b(p^t) = F(p^t) \cap \{v \mid -b\mathbb{1} \le v \le b\mathbb{1}\}$. Then define the following differential inclusion on $\mathbb{R}^m_+$:

$$\frac{dp}{dt} = F_b(p^t). \tag{23}$$

This introduces the possibility that $F_b(p)$ is empty for some p which makes the differential inclusion trivially unsatisfiable. For now we assume that $F_b(p)$ is non-empty in a small neighborhood of p; in the next subsection, we show this assumption is implied by the controllability constraint.

Definition 7.3. *F is* bounded near *p with parameters $\rho > 0$ and $b > 0$ if there exists a ball $B(p, \rho)$ such that for all $q \in B(p, \rho) \cap \mathbb{R}^m_+$, $F_b(q)$ is non-empty and $h''(q)$ is finite.*[15]

A further challenge to applying Theorem 7.2 is that the trajectory may need to start at a location on the boundary, i.e. with one or more prices $p_j = 0$, while F needs to be defined in a ball around p. To resolve this, we extend the definition of F from $\mathbb{R}^m_+$ to $\mathbb{R}^m$, essentially by reflection, but in a way that ensures the trajectory must remain in $\mathbb{R}^m_+$. This then yields the following result.

Lemma 7.3. *Suppose that h is allowable, and F is bounded near p^0 with parameters $\rho > 0$ and $b > 0$. Then there is a time $\bar{t} = \frac{\rho}{b} > 0$ such that* (20) *has an absolutely continuous solution for time interval $[0, \bar{t}]$ with $p(0) = p^0$.*

[15] We remark that this condition is satisfied automatically when $p > \vec{0}$.

7.2.2. Global Existence of a Solution to (20)

By repeatedly applying Lemma 7.3, together with suitable uses of the definition of controllability, the global existence of a solution is guaranteed, as stated in the lemma below.

Lemma 7.4. *Suppose that $h''(p^0)$ is finite, h is allowable and $(\phi, \mathcal{T})$ is controlled. Then for any bounded demand starting price vector p^0 there exists a solution p^t to* (20) *for time range $[0, \infty)$, with p^t an absolutely continuous function for any bounded time span, and $p^t(t = 0) = p^0$.*

7.3. Differential Inclusion (20) *Converges to an Equilibrium*

In Arrow-Debreu economies, it is well-known that if p^* is an equilibrium price vector, then cp^*, where c is any positive constant, is also an equilibrium price vector. It is standard to consider *normalized prices*, price vectors $\hat{p}$ such that $\sum \hat{p} = 1$. Note that for any price vector p with at least one positive price, the corresponding normalized price vector $\hat{p}$ is given by $\hat{p}_j = p_j / (\sum_\ell p_\ell)$.

Lemma 7.5. *Suppose that h is allowable and $h''(p_j^0)$ is finite for all j. Let p^* be any minimizer of ϕ. Then $d_h(p_j^*, p_j^t)$ is finite for all t and j.*

Suppose that ϕ is the potential function for a Fisher economy. Then $\sum_j \frac{d}{dt} d_h(p_j^*; p_j^t) < 0$, unless p^t is a minimizer of ϕ.

Suppose that ϕ is the potential function for an Arrow-Debreu economy. Then let $\hat{p}^*$ be any normalized minimizer of ϕ, and suppose that d_h is the KL-divergence. Then $\hat{p}^t$, the normalized price vector corresponding to p^t, satisfies $\sum_j \frac{d}{dt} d_h(\hat{p}_j^*, \hat{p}_j^t) < 0$, unless p^t is a minimizer of ϕ.

Proof. We prove the result for Fisher economies here; the result for Arrow-Debreu economies is deferred to the appendix.

By Lemma 7.4, p^t is defined for all $t \geq 0$. By Lemma 7.1(i), $h''(p_j^t)$ is finite for all t and j, and hence so is $h'(p_j^t)$.

In the remainder of the proof, to avoid clutter, we write p_j for p_j^t.

As h is always finite, it follows that $d_h(p_j^*, p_j) = h(p_j^*) - h(p_j) - h'(p_j)(p_j^* - p_j)$ is finite. Differentiating gives:

$$\begin{aligned}
\frac{d}{dt} d_h(p_j^*, p_j) &= -\frac{dh(p_j)}{dt} - \frac{dh'(p_j)}{dt}(p_j^* - p_j) + h'(p_j)\frac{dp_j}{dt} \\
&= -h''(p_j) \cdot \frac{dp_j}{dt} \cdot (p_j^* - p_j) \qquad \left(\text{since } \frac{dh(p_j)}{dt} = h'(p_j)\frac{dp_j}{dt}\right) \\
&= \nabla_j \phi(p) \cdot (p_j^* - p_j) \qquad \text{(by Lemma 7.2).}
\end{aligned}$$

By the definition of the subgradient, $\phi(p^*) \geq \phi(p) + \nabla\phi(p) \cdot (p^* - p)$. Thus

$$\sum_j \frac{d}{dt} d_h(p_j^*, p_j) = \sum_j \nabla_j \phi(p) \cdot (p_j^* - p_j) \leq \phi(p^*) - \phi(p) < 0, \tag{24}$$

unless $p = p^*$.

Proof (of Theorem 7.1). In a Fisher economy the prices are always bounded by the maximum of their initial values and $\sum_i e_i$. In an Arrow-Debreu economy, we consider only the normalized prices, and these too are bounded. Let B denote the bounded set of prices. We may assume that B is closed[16]. The proof that follows is for Fisher economies, but it applies to Arrow-Debreu economies too on replacing p by $\hat{p}$ everywhere.

The proof comprises four steps:

[16] If not, replace B by its closure.

1. As p^t lies in a bounded domain, it must have a convergent subsequence, which converges to a point q, say.
2. Let P^* denote the set of equilibrium prices for Fisher economies, or the set of normalized equilibrium prices for Arrow-Debreu economies. Recall that $d_h(p^*, p) = \sum_j d_h(p_j^*, p_j)$. Then, for any fixed $p^* \in P^*$, we can conclude from Lemma 7.5 that $d_h(p^*, p^t)$ is monotonically decreasing. By (21), $d_h(p^*, p^t) \geq 0$; consequently $\lim_{t\to\infty} d_h(p^*, p^t)$ exists, and it must equal $d_h(p^*, q)$, by the continuity of d_h.
3. q is a minimizer of ϕ; i.e. $q \in P^*$. (Proof below.)
4. By the second and the third steps, $d_h(q, p^t) \to d_h(q, q) = 0$. Thus $p^t \to q$. (Proof below.)

Proof of Step 3. Suppose that q were not a minimizer of ϕ.

Note that the set P^* is closed (due to the continuity of ϕ), so $P^* \cap B$ is compact. Let $d(q') = \min_{p' \in P^* \cap B} d_h(p', q')$; since $P^* \cap B$ is compact, the minimum is attained.

Since $q \notin P^*$, as $d(q) = d_h(p^*, q)$ for some $p^* \in P^*$, and as $q \neq p^*$, by (22), $d(q) = d_h(p^*, q) > 0$. Also, by Lemma 7.5, $d_h(p^*, p^t)$ is finite, and hence by continuity of d_h so is $d_h(p^*, q)$; hence $d(q) \leq d_h(p^*, q)$ is also finite. Let $Q = \{q' \mid d(q') \geq d(q)\} \cap B$. Since d_h is continuous and $P^* \cap B$ is compact, it follows that Q is compact. Let $\delta = \min_{q' \in Q} \phi(q') - \phi(p^*)$; since Q is compact, the minimum is attained. By definition, Q contains no minimizer of ϕ, so $\delta > 0$.

From Step 2, for any $p^* \in P^* \cap B$, for all $t \geq 0$, $d_h(p^*, p^t) \geq d_h(p^*, q)$ and $d_h(p^*, q) \geq d(q)$, so $p^t \in Q$ for all $t \geq 0$. By (24), $\frac{d}{dt} d_h(p^*, p^t) \leq -[\phi(p^t) - \phi(p^*)] \leq -\delta < 0$, which implies that $d_h(p^*, p^t)$ will eventually go below zero, a contradiction.

Proof of Step 4. Suppose that p^t does not converge to q. Then there exists an $\epsilon > 0$ such that for any T, there exists a $t(T) > T$ with $||p^{t(T)}, q|| \geq \epsilon$.

Let $A = \{p \mid ||q, p|| \geq \epsilon\}$, which is closed. Note that $A \cap B$ is compact. Since $d_h(q, p)$ is non-negative (but possibly $+\infty$), finite at some $p \in A \cap B$ (e.g. $p^{t(T)}$ for any T), and continuous at every $p \in A \cap B$ at which it is finite, $\inf_{p \in A \cap B} d_h(q, p) = \min_{p \in A \cap B} d_h(q, p) = \delta' > 0$, by (22). Since $p^{t(T)} \in A \cap B$, $d_h(q, p^{t(T)}) \geq \min_{p \in A \cap B} d_h(q, p) = \delta' > 0$, i.e. $d_h(q, p^t)$ does not converge to zero, a contradiction.

8. Further Comments

We have shown that discrete versions of tatonnement converge for Leontief and CES utilities. Here we mention some recent extensions of these results. Cheung [15] has shown the convergence of the discrete tatonnement for economies with CES utilities extends to those with nested CES utilities. Cheung and Cole [17] extended the convergence results in this paper for discrete tatonnement to allow asynchronous updating. In an earlier manuscript [16], they also extended the convergence results to the Ongoing Market model proposed by Cole and Fleischer [24]. In this model, the economy repeats from one time period to the next, and excess demands and supplies are carried forward to successive time periods using finite buffers, which they called warehouses. The purpose of this model was to provide a more natural setting for the tatonnement update process.

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A. Fisher Economies with Leontief Utilities

Proof (of Claim 4.1). This result follows by rewriting e_i as $\sum_k x_{ik}p_k$.

$$\begin{aligned} e_i \sum_\ell \frac{x_{i\ell}}{p_\ell}(\Delta p_\ell)^2 &= \sum_\ell \frac{x_{i\ell}\left(\sum_k x_{ik}p_k\right)}{p_\ell}(\Delta p_\ell)^2 = \sum_{\ell,k} x_{i\ell}x_{ik}\frac{p_k}{p_\ell}(\Delta p_\ell)^2 \\ &= \sum_\ell x_{i\ell}^2(\Delta p_\ell)^2 + \sum_{k,\ell:k\neq\ell} x_{ik}x_{i\ell}\frac{p_k}{p_\ell}(\Delta p_\ell)^2 \\ &= \sum_\ell x_{i\ell}^2(\Delta p_\ell)^2 + \sum_{k<\ell} x_{ik}x_{i\ell}\left(\frac{p_k}{p_\ell}(\Delta p_\ell)^2 + \frac{p_\ell}{p_k}(\Delta p_k)^2\right). \end{aligned}$$

Now, we apply the AM-GM inequality:

$$\begin{aligned} e_i \sum_\ell \frac{x_{i\ell}}{p_\ell}(\Delta p_\ell)^2 &\geq \sum_\ell x_{i\ell}^2(\Delta p_\ell)^2 + \sum_{k<\ell} x_{ik}x_{i\ell}\cdot 2|\Delta p_\ell||\Delta p_k| \\ &= \sum_{j,k} x_{ij}x_{ik}|\Delta p_j||\Delta p_k|. \end{aligned}$$

Proof (of Claim 4.2). We use the bound $\log x \geq x - \frac{11}{18}x^2$ for $|x| \leq \frac{1}{4}$.

$$\begin{aligned} d_{\mathrm{KL}}(p_j+\Delta p_j, p_j) &= (p_j+\Delta p_j)\log(p_j+\Delta p_j) - (p_j+\Delta p_j) - p_j\log p_j + p_j - (\log p_j)\Delta p_j \qquad \text{(by (4) and (5))} \\ &= -\Delta p_j + (p_j+\Delta p_j)\log\left(1+\frac{\Delta p_j}{p_j}\right) \\ &\geq -\Delta p_j + (p_j+\Delta p_j)\left(\frac{\Delta p_j}{p_j} - \frac{11}{18}\frac{(\Delta p_j)^2}{p_j^2}\right) \\ &= \frac{7}{18}\frac{(\Delta p_j)^2}{p_j}\left(1-\frac{11}{7}\frac{\Delta p_j}{p_j}\right) \;\geq\; \frac{7}{18}\frac{17}{28}\frac{(\Delta p_j)^2}{p_j} \;\geq\; \frac{2}{9}\frac{(\Delta p_j)^2}{p_j}. \end{aligned}$$

A.1. Leontief Lower Bound

We prove Theorem 4.2 here.

We consider the following Leontief Fisher economy with two buyers and two goods. Buyer 1 has budget $e_1 = 3$ and $b_{11} : b_{12} = 1 : 3$; buyer 2 has budget $e_2 = 2$ and $b_{21} : b_{22} = 2 : 1$. There is a unique market equilibrium $(p_1^*, p_2^*) = (0, 5)$, with equilibrium demands $(x_{11}^*, x_{12}^*, x_{21}^*, x_{22}^*) = (1/5, 3/5, 4/5, 2/5)$. We will show that if tatonnement starts at a carefully chosen price vector, (p_1, p_2), the potential function value is $\Theta((p_1)^2)$ but in the next time step the potential function drops by only $\Theta((p_1)^3)$.

Let

$$B = \left\{ (p_1,p_2) \;\middle|\; p_1 \leq \bar{\delta} \text{ and } -\frac{2}{5}(p_1)^2 \leq p_1 + p_2 - 5 \leq \frac{2}{5}(p_1)^2 \right\},$$

where $\bar{\delta} > 0$ is a sufficiently small positive number which satisfies several conditions stated in the proofs below.

The price update rule of good j is $p_j^{t+1} = p_j^t \cdot \exp\left(z_j^t/\gamma\right)$.

Lemma A.1. *If a tatonnement starts at a price vector in B, then the prices remain in B throughout the tatonnement.*

Proof. Let (p_1, p_2) be a price vector in B; write $(p_1, p_2) = (\delta, 5 - \delta + C\delta^2)$, where $|C| \leq \frac{2}{5}$. Then the demands are

$$x_1 = \frac{3}{15-2\delta+3C\delta^2} + \frac{4}{5+\delta+C\delta^2} \qquad x_2 = \frac{9}{15-2\delta+3C\delta^2} + \frac{2}{5+\delta+C\delta^2}.$$

Let (p'_1, p'_2) denote the new prices after an update, i.e.

$$p'_1 = \delta \cdot \exp\left(\frac{x_1 - 1}{\gamma}\right) \qquad p'_2 = (5 - \delta + C\delta^2) \cdot \exp\left(\frac{x_2 - 1}{\gamma}\right).$$

The Taylor expansions of x_1, x_2, p'_1, p'_2 (with respect to δ) are

$$x_1 = 1 - \frac{2}{15}\delta + O(\delta^2), \qquad x_2 = 1 + \left(\frac{2}{75} - \frac{C}{5}\right)\delta^2 + O(\delta^3),$$

$$p'_1 = \delta - \frac{2}{15\gamma}\delta^2 + O(\delta^3), \qquad p'_2 = 5 - \delta + \left(C - \frac{C}{\gamma} + \frac{2}{15\gamma}\right)\delta^2 + O(\delta^3).$$

We choose $\bar{\delta}$ to be sufficiently small such that $p'_1 < p_1$. Since $p_1 \le \bar{\delta}$, $p'_1 < \bar{\delta}$.

The Taylor expansion of $\frac{p'_1 + p'_2 - 5}{(p'_1)^2}$ is

$$\frac{p'_1 + p'_2 - 5}{(p'_1)^2} = C\left(1 - \frac{1}{\gamma}\right) + O(\delta).$$

We choose $\bar{\delta}$ to be sufficiently small such that

$$C\left(1 - \frac{1}{\gamma}\right) - \frac{1}{10\gamma} \le \frac{p'_1 + p'_2 - 5}{(p'_1)^2} \le C\left(1 - \frac{1}{\gamma}\right) + \frac{1}{10\gamma}.$$

Since $|C| \le \frac{2}{5}$ and $\gamma \ge 1$, $C\left(1 - \frac{1}{\gamma}\right) - \frac{1}{10\gamma} \ge -\frac{2}{5}$ and $C\left(1 - \frac{1}{\gamma}\right) + \frac{1}{10\gamma} \le \frac{2}{5}$. So (p'_1, p'_2) is in B.

Lemma A.2. *If (p_1^t, p_2^t) is in B, then $\phi(p^t) - \phi(p^{t+1}) = O((p_1)^3)$ and $\phi(p^t) - \phi(p^*) = \Theta((p_1)^2)$.*

PROOF. Write $(p_1^t, p_2^t) = (\delta, 5 - \delta + C\delta^2)$, where $|C| \le \frac{2}{5}$. Since ϕ is convex,

$$\begin{aligned}
\phi(p^t) - \phi(p^{t+1}) &\le -\nabla\phi(p^t) \cdot \left(p^{t+1} - p^t\right) \\
&= (x_1 - 1)\left(\exp\left(\frac{x_1 - 1}{\gamma}\right) - 1\right)p_1 + (x_2 - 1)\left(\exp\left(\frac{x_2 - 1}{\gamma}\right) - 1\right)p_2 \\
&= O\left(\frac{p_1(x_1 - 1)^2}{\gamma}\right) + O\left(\frac{p_2(x_2 - 1)^2}{\gamma}\right).
\end{aligned}$$

Recall the Taylor expansions of x_1 and x_2 in the proof of Lemma A.1. We choose $\bar{\delta}$ to be sufficiently small so that

$$|x_1 - 1| = O(\delta), \qquad |x_2 - 1| = O(\delta^2).$$

Then

$$\phi(p^t) - \phi(p^{t+1}) \le O(\delta) \cdot O\left(\frac{\delta^2}{\gamma}\right) + O(1) \cdot O\left(\frac{\delta^4}{\gamma}\right) = \frac{1}{\gamma}O(\delta^3).$$

Next, we will show that $\phi(p) - \phi(p^*)$ is $\Theta(\delta^2)$. Let $\Delta^* p_\ell = p^*_\ell - p_\ell$. Note that (see the proof of Lemma 4.1)

$$\phi(p^*) - \ell_\phi(p^*; p) = \sum_j x_j \Delta^* p_j - \sum_i e_i \log\left(1 + \frac{\sum_\ell b_{i\ell}\Delta^* p_\ell}{\sum_\ell b_{i\ell} p_\ell}\right).$$

We choose $\bar{\delta}$ to be sufficiently small such that for $i = 1, 2$, $\left|\frac{\sum_\ell b_{i\ell}\Delta^* p_\ell}{\sum_\ell b_{i\ell} p_\ell}\right| \le \frac{1}{4}$. Then applying (16) yields

$$\begin{aligned}
\phi(p^*) - \ell_\phi(p^*; p) &\le \sum_j x_j \Delta^* p_j - \sum_i e_i \left[\frac{\sum_\ell b_{i\ell}\Delta^* p_\ell}{\sum_\ell b_{i\ell} p_\ell} - \frac{2}{3}\left(\frac{\sum_\ell b_{i\ell}\Delta^* p_\ell}{\sum_\ell b_{i\ell} p_\ell}\right)^2\right] \\
&= \frac{2}{3}\sum_i e_i \left(\sum_\ell \frac{x_{i\ell}}{e_i}\Delta^* p_\ell\right)^2 = \frac{2}{3}\sum_i \frac{1}{e_i}\left(\sum_\ell x_{i\ell}\Delta^* p_\ell\right)^2.
\end{aligned}$$

Note that $\Delta^* p_1 = -\delta$ and $\Delta^* p_2 = \delta - C\delta^2$. The Taylor expansions of the $\{x_{ij}\}$ are

$$x_{11} = \frac{1}{5} + O(\delta) \qquad x_{12} = \frac{3}{5} + O(\delta) \qquad x_{21} = \frac{4}{5} + O(\delta) \qquad x_{22} = \frac{2}{5} + O(\delta).$$

Hence, the Taylor expansions of $\frac{1}{e_i}\left(\sum_\ell x_{i\ell}\Delta^* p_\ell\right)^2$ are

$$\begin{aligned}\frac{1}{e_1}\left(x_{11}\Delta^* p_1 + x_{12}\Delta^* p_2\right)^2 &= \frac{4}{75}\delta^2 + O(\delta^3),\\ \frac{1}{e_2}\left(x_{21}\Delta^* p_1 + x_{22}\Delta^* p_2\right)^2 &= \frac{2}{25}\delta^2 + O(\delta^3).\end{aligned}$$

Thus

$$\phi(p^*) - \ell_\phi(p^*; p) \le \frac{4}{45}\delta^2 + O(\delta^3).$$

Then

$$\begin{aligned}\phi(p^*) - \phi(p) &\le \frac{4}{45}\delta^2 + O(\delta^3) - z_1\Delta^* p_1 - z_2\Delta^* p_2\\ &= \frac{4}{45}\delta^2 - \left(-\frac{2}{15}\delta\right)(-\delta) - \left(\frac{2}{75} - \frac{C}{5}\right)\delta^2 \cdot (\delta - C\delta^2) + O(\delta^3)\\ &= -\frac{2}{45}\delta^2 + O(\delta^3).\end{aligned}$$

We choose $\bar{\delta}$ to be sufficiently small such that $\phi(p) - \phi(p^*) = \Theta(\delta^2)$.

Proof. (of Theorem 4.2.) If the tatonnement starts at $(\bar{\delta}, 5 - \bar{\delta})$, by Lemma A.1, the prices stay in B throughout the tatonnement. Then by Lemma A.2, at every time step,

$$\phi(p^t) - \phi(p^{t+1}) = r_j^c \cdot \Theta\left(\phi(p^t) - \phi(p^*)\right)^{3/2}.$$

This easily yields

$$\phi(p^t) - \phi(p^*) = \Omega\left(\frac{\phi(p^0) - \phi(p^*)}{t^2}\right).$$

B. Fisher Economies with Complementary CES Utilities

Lemma 5.5 is a subcase of the following lemma, which includes a similar result for continuous tatonnement.

Lemma B.1. *Let $\overline{U} = U$ for any continuous tatonnement, and let $\overline{U} = 2U$ for the discrete tatonnement with update rule (14). For any continuous tatonnement and for all t, $p_j^*/p_j^t \le \max\{p_j^*/p_j^\circ, (L^*/\overline{U})^{\min_i \rho_i}\}$. For the discrete tatonnement and for all t, $p_j^*/p_j^t \le 2 \cdot \max\{p_j^*/p_j^\circ, (L^*/\overline{U})^{\min_i \rho_i}\}$.*

Proof. The lemma follows from the following two observations.
Observation 1. No price will exceed $\overline{U}$ during the entire tatonnement.
Reason. Suppose not, then let $t = \tau$ be the first time when some price, say p_k, exceed $\overline{U}$. Then $p_k^\tau \ge M$ and $x_k^\tau \le M/p_k^\tau \le 1$. In the continuous tatonnement, the price update rule will not increase p_k any further.

For the discrete tatonnement we argue as follows. At $t = \tau - 1$, $p_k^{\tau-1} < \overline{U} = 2U$. However, as p_k can at most double in one time unit, $p_k^{\tau-1} \ge U$, and hence $p_k^{\tau-1} \ge M$. By the same argument as for x_k^τ, $x_k^{\tau-1} \le 1$. By the price update rule, $p_k^\tau \le p_k^{\tau-1} < \overline{U}$, a contradiction.

Observation 2. $p_k \ge \min\{p_k^\circ, (\overline{U}/L^*)^{\min_i \rho_i} p_k^*\}$ throughout the entire continuous tatonnement process, and half this value in the discrete case.

Reason. Suppose that for some k, $p_k \le L^*(\overline{U}/L^*)^{\min_i \rho_i} p_k^*$. We claim that $x_k \ge 1$.

At equilibrium prices, all demands equal 1. If the prices are all raised by a factor of $\frac{\overline{U}}{L^*}$, then all demands equal $\frac{L^*}{\overline{U}}$. Note that now all prices are at least $\overline{U}$.

Now reduce the price of p_k from $\frac{\overline{U}}{L^*} p_k^*$ to $\left(\frac{\overline{U}}{L^*}\right)^{\min_i \rho_i} p_k^*$, that is, reduce the price by a factor of $\left(\frac{\overline{U}}{L^*}\right)^{1-\min_i \rho_i}$. Recall (13); note that the price reduction can only decrease S_i. Then, the new demand x_k' for good k is bounded as follows:

$$x_k' \ge \frac{L^*}{\overline{U}} \cdot \left[\left(\frac{\overline{U}}{L^*}\right)^{1-\min_i \rho_i}\right]^{1-\max_i c_i} = \frac{L^*}{\overline{U}} \cdot \left[\left(\frac{\overline{U}}{L^*}\right)^{1-\min_i \rho_i}\right]^{1/(1-\min_i \rho_i)} = 1.$$

We just proved that when $p_k = \left(\frac{\overline{U}}{L^*}\right)^{\min_i \rho_i} p_k^*$ and all other prices are at specified values which are all at least $\overline{U}$, the demand for good k is at least 1. By Observation 1, no price exceeds $\overline{U}$ during the entire tatonnement process. In complementary economies, since the demand for one good increases when the prices of other goods decrease, we have shown that $x_k \ge 1$ if $p_k \le \left(\frac{\overline{U}}{L^*}\right)^{\min_i \rho_i} p_k^*$.

In the case of the continuous tatonnement, it follows that no price can decrease below the minimum of this value and the initial value of this price.

For the discrete case, we argue as follows. Let

$$\bar{L}_k = (1/2) \cdot \min\{p_k^\circ, (\overline{U}/L^*)^{\min_i \rho_i} p_k^*\}.$$

Suppose that Observation 2 were incorrect, then let $t = \tau$ be the first time when some price, say p_j, is below $\bar{L}_j$. At $t = \tau - 1$, $p_j^{\tau-1} \ge \bar{L}_j$. However, as p_j can reduce by at most half in one time unit, $p_j^{\tau-1} \le 2\bar{L}_j$. Then $x_j^{\tau-1} \ge 1$. By the price update rule, $p_j^\tau \ge p_j^{\tau-1} > \bar{L}_j$, a contradiction.

C. Fisher Economies with Substitute CES Utilities

In substitute CES utilities, $0 < \rho < 1$, so $c = \rho/(\rho-1)$ is negative; by contrast, the parameter c for complementary CES utilities is positive. Recall that $c_{\min} = \min_i c_i$.

C.1. The Upper Bound: Good Progress on a Price Update

Lemma C.1. *Suppose that for all j, $|\Delta p_j|/p_j \le \min\{1/4, 1/|c_{\min}|\}$. Then*

$$\phi(p+\Delta p) - \ell_\phi(p+\Delta p; p) \le (1+|c_{\min}|) \sum_j \frac{x_j}{p_j} (\Delta p_j)^2.$$

Proof. We will use the following bound, which follows from a simple power series expansion: if c is negative and $|\Delta p_j|/p_j \le \min\{1/4, 1/|c|\}$, then

$$(p_j + \Delta p_j)^c \le p_j^c + c p_j^{c-1}(\Delta p_j) + c(c-1) p_j^{c-2} (\Delta p_j)^2. \tag{C.1}$$

Recall from the proof of Lemma 5.3 that

$$D_\phi = \phi(p+\Delta p) - \ell_\phi(p+\Delta p; p) = \sum_j x_j \Delta p_j + \sum_i \frac{e_i}{|c_i|} \log\left(\frac{\sum_\ell b_{i\ell}(p_\ell + \Delta p_\ell)^{c_i}}{S_i(p)}\right).$$

We apply (C.1) and the simple bound $\log(1+\epsilon) \leq \epsilon$ for $\epsilon \geq -1$ to yield

$$\begin{aligned}
D_\phi &\leq \sum_j x_j \Delta p_j + \sum_i \frac{e_i}{|c_i|} \log\left(\frac{\sum_\ell b_{i\ell}\left(p_\ell^{c_i} - |c_i| p_\ell^{c_i-1}\Delta p_\ell + |c_i|(|c_i|+1) p_\ell^{c_i-2}(\Delta p_\ell)^2\right)}{S_i(p)} \right) \\
&= \sum_j x_j \Delta p_j + \sum_i \frac{e_i}{|c_i|} \log\left(1 - |c_i| \sum_\ell \frac{b_{i\ell} p_\ell^{c_i-1}}{S_i(p)} \Delta p_\ell + |c_i|(|c_i|+1) \sum_\ell \frac{b_{i\ell} p_\ell^{c_i-2}}{S_i(p)} (\Delta p_\ell)^2 \right) \\
&\leq \sum_j x_j \Delta p_j + \sum_i \frac{e_i}{|c_i|} \left(-|c_i| \sum_\ell \frac{x_{i\ell}}{e_i} \Delta p_\ell + |c_i|(|c_i|+1) \sum_\ell \frac{x_{i\ell}}{e_i p_\ell} (\Delta p_\ell)^2 \right) \\
&\leq \sum_j x_j \Delta p_j - \sum_\ell x_\ell \Delta p_\ell + (1+|c_{\min}|) \sum_\ell \frac{x_\ell}{p_\ell} (\Delta p_\ell)^2 \\
&= (1+|c_{\min}|) \sum_j \frac{x_j}{p_j} (\Delta p_j)^2.
\end{aligned}$$

To allow us to apply Lemma C.1 in our analysis, we will require that $\gamma_j^t = 5(1+|c_{\min}|) \cdot \max\{1, x_j^t\}$.

Lemma C.2. *Suppose that* $|p_j^{t+1} - p_j^t| \leq \min\{1/4, 1/|c_{\min}|\} \cdot p_j^t$ *for all* j. *Then*

$$\phi(p^t) - \phi(p^{t+1}) \geq \frac{1}{2} \sum_j \frac{z_j^2 p_j^t}{\gamma_j^t}.$$

Proof. It is almost identical to the proof of Lemma 5.1. It uses the bound from Lemma C.1 instead of the bound from Lemma 5.3. This changes the factor 2 in (15) to $(1+|c_{\min}|)$, and then the new value for γ_j yields (**??**). The rest of the proof is identical.

C.2. An Upper Bound on the Distance to Equilibrium

We will need the following bound. Recall that we are letting log denote the natural logarithm.

Claim C.1. *Let* $\bar{r}$ *be a fixed number greater than* 0. *If* $-1 < x \leq \bar{r} - 1$, *then*

$$\log(1+x) \leq x - \frac{\bar{r} - 1 - \log \bar{r}}{(\bar{r}-1)^2} x^2. \tag{C.2}$$

Proof. Consider the function $[x - \log(1+x)]/x^2$ in the domain $(-1, +\infty)$; at $x = 0$, the function takes value $1/2$, so that the function is continuous at $x = 0$. The claim follows by simply noting that this function is a decreasing function for $x \in (-1, +\infty)$.

Lemma C.3. *Suppose that* $r_j^t = p_j^*/p_j \leq \bar{r}_j$ *for all* j. *Then*

$$\phi(p^*) - \ell_\phi(p^*; p) \geq \sum_j \frac{\bar{r}_j - 1 - \log \bar{r}_j}{(\bar{r}_j - 1)^2} x_j \frac{(p_j^* - p_j)^2}{p_j}.$$

Proof. Let $s_{ik} = x_{ik} p_k$. First,

$$\begin{aligned}
\phi(p + \Delta^* p) - \ell_\phi(p + \Delta^* p; p) &= \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i} \log\left(\sum_k \frac{b_{ik}(p_k)^{c_i}}{\sum_\ell b_{i\ell}(p_\ell)^{c_i}} \left(1 + \frac{\Delta^* p_k}{p_k}\right)^{c_i} \right) \\
&= \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i} \log\left(\sum_k \frac{s_{ik}}{e_i} \left(1 + \frac{\Delta^* p_k}{p_k}\right)^{c_i} \right).
\end{aligned}$$

Note that $\sum_k \frac{s_{ik}}{e_i} = 1$. Thus, by the concavity of the log function,
$\log\left(\sum_k \frac{s_{ik}}{e_i}\left(1+\frac{\Delta^* p_k}{p_k}\right)^{c_i}\right) \geq \sum_k \frac{e_{ik}}{e_i}\log\left(1+\frac{\Delta^* p_k}{p_k}\right)^{c_i}$. Then

$$\begin{aligned}
\phi(p^*) - \ell_\phi(p^*; p) &\geq \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i}\sum_k \frac{s_{ik}c_i}{e_i}\log\left(1+\frac{\Delta^* p_k}{p_k}\right) \\
&= \sum_j x_j \Delta^* p_j - \sum_k x_k p_k \log\left(1+\frac{\Delta^* p_k}{p_k}\right) \\
&\geq \sum_j x_j \Delta^* p_j - \sum_k x_k p_k \left(\frac{\Delta^* p_k}{p_k} - \frac{\bar{r}_k - 1 - \log \bar{r}_k}{(\bar{r}_k - 1)^2}\left(\frac{\Delta^* p_k}{p_k}\right)^2\right) \qquad \text{(By Claim C.1)} \\
&= \sum_j \frac{\bar{r}_j - 1 - \log \bar{r}_j}{(\bar{r}_j - 1)^2} x_j \frac{(\Delta^* p_j)^2}{p_j}.
\end{aligned}$$

Lemma C.4. *Suppose that $r_j^t = p_j^*/p_j^t \leq \bar{r}_j$. Then*

$$\phi(p^t) - \phi(p^*) \leq (1 + |c_{\min}|) \cdot \max_j \left\{10, \frac{5(\bar{r}_j - 1)^2}{2(\bar{r}_j - 1 - \log \bar{r}_j)}\right\} \cdot \sum_j \frac{z_j^2 p_j^t}{\gamma_j^t}.$$

Proof. The proof is almost identical to the proof of Lemma 5.2. The ratio $(\bar{r}_j - 1 - \log \bar{r}_j)/(\bar{r}_j - 1)^2$ from Lemma C.3 plays the role of $H_c(\bar{r}_j)$ in Lemma 5.4.

In the proof of Lemma 5.2, we used the fact that $\gamma_j = 5$ in Case 1, and that $\gamma_j = 5 \cdot \max\{1, x_j\}$ in Case 2. Here, we replace the number 5 in both cases with $5(1 + |c_{\min}|)$, yielding the new bound.

Proof (of Theorem 6.1). This is almost identical to the proof of Theorem 5.1. We first note that Cole and Fleisher [24] showed that r_j^t remains bounded by some finite number $\bar{r}_j$ throughout the tatonnement process. Instead of the bounds from Lemmas 5.1 and 5.2, we use the bounds from Lemmas C.2 and C.4. This gives

$$\phi(p^{t+1}) - \phi(p^*) \leq [\phi(p^t) - \phi(p^*)]\left[1 - \frac{1}{2(1 + |c_{\min}|)}\left(\max_j\left\{10, \frac{5(\bar{r}_j - 1)^2}{2(\bar{r}_j - 1 - \log \bar{r}_j)}\right\}\right)^{-1}\right].$$

Noting that $1 + |c_{\min}|$ and the max's two arguments are finite, we are done.

D. Continuous Time Tatonnement

D.1. Differential Inclusion and Semi-Continuity of Sets

The following definitions are taken from Smirnov's text [62].

Definition D.1. *A differential inclusion is an equation of the form $\frac{dp}{dt} \in F(t, p(t))$, where $F(t, p)$ is a non-empty set for all t and p. This generalizes standard differential equations of the form $\frac{dp}{dt} = f(t, p(t))$, where $f(t, p)$ is single-valued.*

In our setting, F is a function of p alone.

Let $\mathbb{P}(A)$ denote the power set of the set A. Let $\Omega(a)$ denote an open neighbourhood of a point a.

Definition D.2. *A set-valued map $F : Z \to \mathbb{P}(Y)$ is* upper semi-continuous *at $z_0 \in Z$ if for any open set $M \in \mathbb{P}(Y)$ which contains $F(z_0)$, there exists $\Omega(z_0)$ such that for all $z \in \Omega(z_0)$, $F(z) \subset M$. A set-valued map F is upper semi-continuous if it is so at every $z_0 \in Z$.*

A set-valued map $F : Z \to \mathbb{P}(Y)$ is lower semi-continuous *at $z_0 \in Z$ if for any $y_0 \in F(z_0)$ and any neighbourhood $\Omega(y_0)$, there exists a neighborhood $\Omega(z_0)$ such that for all $z \in \Omega(z_0)$, $F(z) \cap \Omega(y_0) \neq \emptyset$. A set-valued map F is lower semi-continuous if it is so at every point $z_0 \in Z$.*

A set-valued map $F : Z \to \mathbb{P}(Y)$ is continuous *at $z_0 \in Z$ if it is both upper and lower semi-continuous at z_0. A set-valued map F is continuous if it is so at every $z_0 \in Z$.*

The following well-known Maximum Theorem provides results on set-valued map semi-continuity, which are among the required conditions for the existence of a solution to our differential inclusions.

Theorem D.1 (Maximum Theorem, [5, p. 116]). *Let $u : P \times X \to \mathbb{R}$ be a continuous function, and $C : P \to \mathbb{P}(X)$ be a compact set-valued map. Let $C^*(p) = \arg\max_{x \in C(p)} u(p,x)$ and $u^*(p) = \max_{x \in C(p)} u(p,x)$. If C is continuous at some p, then u^* is continuous at p and C^* is non-empty, compact and upper semi-continuous at p.*

In our scenarios, X is the set of bundles of goods and u is the utility function of an agent. C maps a price vector to the set of affordable bundles; note that C is a compact set-valued map unless the price vector contains a zero price.

For any sets $A_1, A_2, \cdots, A_k$, let their *sumset* be $\left\{\sum_{i=1}^{k} a_i \mid a_i \in A_i\right\}$. We state the following basic facts, which will be useful later.

Lemma D.1. *(a) If $A_1, A_2, \cdots, A_k$ are convex and compact, then their sumset is convex and compact.*
(b) If $A_1, A_2, \cdots, A_k : Z \to \mathbb{P}(Y)$ are upper semi-continuous at $z \in Z$, then their sumset is upper semi-continuous at z.
(c) If $F_1, F_2 : Z \to \mathbb{P}(Y)$ are two set-valued maps which are upper semi-continuous at $z \in Z$, the map $F^{\cap} : Z \to \mathbb{P}(Y)$, defined as $F^{\cap}(z) = F_1(z) \cap F_2(z)$, is also upper semi-continuous at $z \in Z$.

D.2. Existence of a Solution

Proof (of Lemma 7.2.). The minimizer in (18) must have a zero derivative:

$$\nabla_j \phi(p^t) + \tfrac{1}{\epsilon} \frac{d\big(d_h(p_j, p_j^t)\big)}{dp_j} = 0. \tag{D.1}$$

Since $\frac{d\left(d_h(p_j, p_j^t)\right)}{dp_j} = h'(p_j) - h'(p_j^t)$, substituting in (D.1) and solving for p_j gives

$$p_j(\epsilon) = h'^{-1}\left(h'(p_j^t) - \epsilon \nabla_j \phi(p^t)\right).$$

Note that since h is strictly convex, h' is strictly increasing and hence is invertible. For notational convenience, let $g(y) = h'^{-1}(y)$. Then $h'(g(y)) = y$, $h''(g(y)) \cdot g'(y) = 1$, therefore $g'(p_j) = \frac{1}{h''(g(p_j))}$. Also note that $g(h'(y)) = y$. Using these we obtain

$$g'(h'(p_j)) = \frac{1}{h''(p_j)}. \tag{D.2}$$

Strictly speaking, the above argument is not valid for $p_j = 0$ if $h'(0) = -\infty$. But in this case, we can check directly that (D.2) is still correct, for then $g'(-\infty) = 0$ and $h''(0) = \infty$.

$$\begin{aligned} \text{Now,} \quad \lim_{\epsilon \to 0} \frac{p_j(\epsilon) - p_j^t}{\epsilon} &= \lim_{\epsilon \to 0} \frac{g(h'(p_j^t) - \epsilon \nabla_j \phi(p^t)) - g(h'(p_j^t))}{\epsilon} \\ &= -g'(h'(p_j^t)) \cdot \nabla_j \phi(p^t) \\ &= -\nabla_j \phi(p^t)/h''(p_j^t) \quad \text{(by (D.2))}. \end{aligned}$$

D.2.1. Missing Proofs in Section 7.1.1

Proof (of Lemma 7.1.). We begin with (i). If the economy is a Fisher economy then prices remain bounded by the maximum of their initial value and the amount of money in the economy. So suppose the economy is not a Fisher economy; then, by assumption, $\int_p^\infty h''(q)dq = \infty$ for all $p > 0$. Let $p_{\max} = \max p_j$. Define $M^t = \sum_j p_j^t \leq p_{\max} \cdot m$. Then $z_{\max} \leq m$. So $\frac{d}{dt} p_{\max}^t \leq m/h''(p_{\max}^t)$.

Let $\bar{t}$ be the earliest time at which $p^t_{\max}$ could be infinite. Let $\text{tmin} = \arg\min_{t<\bar{t}} p^t_{\max}$. If $p^{\text{tmin}}_{\max} > 0$, then by Condition A2,

$$\bar{t} \geq \frac{1}{m} \int_{p^{\text{tmin}}_{\max}}^{\infty} h''(p)dp = \infty,$$

and if $p^{\text{tmin}}_{\max} = 0$, then the same bound holds by Conditions A2 and B2.

Now we show (ii). If condition B1 of Definition 7.1 holds then the result is immediate. So suppose that condition B2 holds. By assumption, $h''(p) = \infty$ only if $p = 0$. As $z_j \geq -1$ always, $\nabla_j \phi(p) \leq 1$ always. Consequently, by Lemma 7.2, $\frac{dp^t_j}{dt} \geq -1/h''(p^t_j)$. Suppose that $p^0_j > 0$. Then let $\bar{t} > 0$ be the earliest time at which p_j could be zero. We use condition B2 to justify the last equality below:

$$\bar{t} \geq -\int_0^{p^0_j} \frac{dp^t_j}{dp^t_j/dt} \geq \int_0^{p^0_j} h''(p)dp = \infty.$$

Thus only at time $t = \infty$ can p_j be 0, and hence only at time $t = \infty$ can $h''(p_j)$ be ∞.

Example D.2. *Consider an Arrow-Debreu economy with one agent and two goods. The agent has one unit of each good as initial endowment. The agent wants only good 1. So the equilibrium price vector is $(p^*_1, p^*_2) = (p, 0)$ for any $p > 0$. At any (p_1, p_2), the excess demand for good 1 is $(p_1 + p_2)/p_1 - 1 = p_2/p_1$ and the excess demand for good 2 is -1.*

Suppose the tatonnement starts at $(p_1, p_2) = (2, 1)$ and h satisfies $h''(p) = 1/p$ for $p \leq 1$ and $h''(p) = 1/p^3$ for $p \geq 1$. Then $\frac{dp^t_2}{dt} = -p^t_2$ and $\frac{dp^t_1}{dt} = (p^t_1)^2 p^t_2$. The solution is $p_1(t) = \frac{2}{2e^{-t}-1}$ and $p_2(t) = e^{-t}$. Note that $p_1(t)$ blows up to $+\infty$ at $t = \log 2$.

D.2.2. Missing Proofs in Section 7.1.2

Claim D.1. *Fisher economies with CES, Leontief and linear utilities along with* any *tatonnement rule are all controlled.*

Proof. We first observe that in Fisher economy prices remain bounded. The following notation will be helpful. Let U be the maximum initial price and M the total money in the economy, and let $\overline{U} = \max\{U, M\}$. Observe that for any j, if $p_j = \overline{U}$, then $x_j \leq 1$, and consequently any tatonnement rule will not increase p_j beyond $\overline{U}$.

We can now show that for Fisher economies $1/h''$ remains bounded. For $h'' > 0$ and consequently in the bounded region $\mathbb{R}^m_+ \cap \{p \leq \overline{U}\,\mathbb{1}\}$ the supremum of $1/h''$ is its maximum, which is therefore finite.

Thus to prove the result of the lemma it suffices to show that $-\nabla_j \phi(p) = z_j(p)$ remains bounded throughout the tatonnement.

We begin by considering substitutes CES utilities. Let $f = \min_j\{p_j/p^*_j, 1\}$. Cole and Fleischer [24] showed that if $p_j = f\,p^*_j$, then $x_j \geq 1$. Thus if p_j is ever reduced to $f\,p^*_j$, the tatonnement update will not decrease it further. Consequently, for all j, $p_j \geq fp^*_j$ throughout the tatonnement process. Hence $x_j \leq M/(fp^*_j)$ throughout the tatonnement process, for all j, where M is the total money in the economy. It follows that $z_j \leq M/(fp^*_j) - 1$, for all j. This analysis applies to linear utilities too.

We turn to complementary CES utilities. By Lemma B.1, $p^t_j \geq p^*_j \cdot \min\{p^0_j/p^*_j, (\overline{U}/L^*)^{\min_i \rho_i}\}$, where $L^* = \min_j\{p^*_j\}$. It follows that the demands are upper bounded by $x_j \leq \max\{p^*_j/p^0_j, (L^*/\overline{U})^{\min_i \rho_i}\}$, and hence $z_j \leq \max\{p^*_j/p^0_j, (L^*/\overline{U})^{\min_i \rho_i}\} - 1$.

Finally, we consider Leontief utilities. By Lemma 4.2,[17] $x^t_j \leq x^\circ_j + \sum_i \max_{k:b_{ik}>0} \frac{b_{ij}}{b_{ik}}$, and hence $z^t_j \leq z^\circ_j + \sum_i \max_{k:b_{ik}>0} \frac{b_{ij}}{b_{ik}}$.

Claim D.2. *If h is controlling then $(\phi, \mathcal{T})$ is controlled.*

[17] Lemma 4.2 handles discrete tatonnement, but the same proof can be reused for continuous tatonnement, and the factor 2 in the upper bound can be saved.

Proof. As h is controlling, in finite time $\bar{t}$, the trajectory is both upper-bounded and bounded away from zero[18], say $0 < \underline{p}(\bar{t}) \leq p_j^t \leq \bar{p}(\bar{t}) < +\infty$, for all j and for all $0 \leq t \leq \bar{t}$. Then there exists a neighborhood Ω of the trajectory up to time $\bar{t}$ such that all prices in Ω are between $\underline{p}(\bar{t})/2$ and $\bar{p}(\bar{t}) + 1$. Set $c(p^0, \bar{t}) = \bar{p}(\bar{t}) + 1$.

For all $p \in \Omega$, $0 < \underline{p}(\bar{t})/2 \leq p_j \leq \bar{p}(\bar{t}) + 1 < +\infty$, so $h''(p_j)$ is bounded away from 0.

As ϕ is convex, $\nabla\phi$ is finite except possibly at the boundary, i.e. when one or more prices is zero. When all prices are between $\underline{p}(\bar{t})/2$ and $\bar{p}(\bar{t}) + 1$, $\nabla\phi$ is bounded. Combined with the last paragraph, $|-\nabla\phi(p)/h''(p_j)|$ is bounded on Ω. Set $b(p^0, \bar{t})$ to be an upper bound of $|-\nabla\phi(p)/h''(p_j)|$ on Ω.

D.2.3. Local Existence: Proof of Lemma 7.3

To prove Lemma 7.3, we need the intermediate lemmas D.2–D.4 below.

Lemma D.2. *Suppose that h is allowable and that F is bounded near p. Then $F_b(p)$ is convex-valued, compact-valued and upper semi-continuous at p.*

Proof. Let $\Omega(p)$ be the neighborhood of p given by the assumption that F is bounded near p (Definition 7.3), and let $B \subset \mathbb{R}_+^m$ be a compact neighborhood of p such that $B \subset \Omega(p)$ and every positive price in p is positive in B. By our choice of B $h''(q_j)$ is positive and finite for all $q \in B$ and for all j, so there exists a positive number $\bar{h}$ such that $h''(q_j) \leq \bar{h}$ for all $q \in B$ and for all j. Then on B, $b \geq |z_j(q)/h''(q_j)| \geq |z_j(q)/\bar{h}|$, i.e. $x_j(q) = z_j(q) + 1 \leq b\bar{h} + 1$. Let $\bar{b}$ denote $b\bar{h} + 1$.

We apply Theorem D.1 with $P = \Omega(p)$, $X = \left[0, \bar{b}\right]^m$. u is the utility function of an agent, which we assume to be continuous and concave. For any $q \in \Omega(p)$, $C(q)$ is the set of all affordable bundles in X of the agent at price q. It is well known that $C(q)$ is continuous, and since its range is confined to the compact set X, $C(q)$ is compact-valued. By Theorem D.1, $C^*(p)$, the set of all affordable optimal bundles of the agent at price p contained in X, is compact and upper semi-continuous at p. By our assumption that $F_b(p)$ is non-empty, $C^*(p)$ is also a subset of all affordable optimal bundles of the agent at price p globally (i.e. without confinement to X). Also, since u is concave, $C^*(p)$ is convex.

By the definition of $C^*(p)$ and ϕ, $-\nabla\phi(p)$ is the sumset of $C^*(p)$ over all agents and the set $\{-\mathbb{1}\}$. As $C^*(p)$ is non-empty for each agent, $-\nabla\phi(p)$ is also non-empty. By Lemma D.1(a) and (b), $-\nabla\phi(p)$ is convex and compact, and it is upper semi-continuous at any p. $(F_b)_j$ is $-\nabla_j\phi(p)$ divided by $h''(p_j)$, while $1/h''$ is continuous and positive at any $p \in P$. So the division by h'' will not affect convexity, compactness and upper semi-continuity.

The following corollary is immediate.

Corollary D.1. *Any solution to system* (23) *over time interval* $[0, \bar{t}]$ *starting at a price vector* p^0 *such that* F *is bounded near* p^0 *is also a solution to system* (20).

As discussed previously, we want to extend the domain for the differential inclusion to $\mathbb{R}^m$. We will work with F_b rather than F, however. To help specify the new differential inclusion system, for any price vector p, we introduce the following notation: letting $p = (p_j)$, we define $p_j^+ = \max\{0, p_j\}$ and $p^+ = (p_j^+)$. The new system is given by

$$\frac{dp^t}{dt} \in G_b(p^t), \tag{D.3}$$

with G_b defined as follows:

1. For $p \in \mathbb{R}_+^m$, $G_b(p) = F_b(p)$.
2. For $p \notin \mathbb{R}_+^m$, let $J(p) = \{j \mid p_j < 0\}$, then set $G_b(p) = G_b(p^+) \cap \{z \mid \forall j \in J(p),\, z_j \geq 0\}$.

Lemma D.3. *Let $p \in \mathbb{R}^m$. Suppose that h is allowable and F is bounded near p^+. Then $G_b(p)$ is convex, compact and upper semi-continuous at p.*

[18] These follow easily from the proofs of Lemma 7.1(i) and (ii).

Proof. As $G_b \equiv F_b$ in $\mathbb{R}^m_+$, by Lemma D.2, the result is immediate for $p > \vec{0}$.

For the other p's, note that $G_b(p) = G_b(p^+) \cap \{z \mid \forall j \in J(p),\, z_j \geq 0\}$ is the intersection of two sets, the first being convex and compact and the second being convex and closed. So $G_b(p)$ is convex and compact. What remains is to check upper semi-continuity at these p's. There are two cases: $p \in \mathbb{R}^m_+$ but it has some zero prices, or $p \notin \mathbb{R}^m_+$.

Case 1: $p \in \mathbb{R}^m_+$ but it has some zero prices. For any open set M which contains $G_b(p) = F_b(p)$, by Lemma D.2, we can take a sufficiently small neighborhood $B(p, \delta)$ of p such that for all $q \in B(p, \delta) \cap \mathbb{R}^m_+$, $F_b(q) \subset M$. Then, for any $q \in B(p, \delta) \setminus \mathbb{R}^m_+$, note that $q^+ \in B(p, \delta)$ since $||q^+, p|| \leq ||q, p||$, and, of course, $q^+ \in \mathbb{R}^m_+$. Thus $F_b(q^+) \subset M$; and $G_b(q) \subseteq G_b(q^+) = F_b(q^+) \subset M$. So G_b is upper semi-continuous at p.

Case 2: $p \notin \mathbb{R}^m_+$. For any $q \in \mathbb{R}^m$, let $V(q)$ denote the set $\{v \mid \forall j \in J(q),\, v_j \geq 0\}$. For any $q \notin \mathbb{R}^m_+$, $G_b(q) = G_b(q^+) \cap V(q)$. By Case 1 and our conditions on p, $G_b(p^+)$ is upper semi-continuous at p^+. p^+ is continuous in p. Hence $G_b(p^+)$ is upper semi-continuous at p. Next, we observe that there exists a small $\delta > 0$ such that for all $q \in B(p, \delta)$, if $p_j \neq 0$, then $\mathrm{sign}(q_j) = \mathrm{sign}(p_j)$ and consequently $V(q) \subseteq V(p)$; it immediately follows that $V(p)$ is upper semi-continuous at p. Now, by Lemma D.1(c), $G_b(p)$ is upper semi-continuous at p.

Lemma D.4. *Any solution to system* (D.3) *over time interval* $[0, \bar{t}]$ *starting at price vector* p^0 *is also a solution of* (20) *if* F *is bounded near* p^0.

Proof. We will show that any solution of (D.3) is a solution of (23). The result then follows from Lemma D.1.

In the definition of G_b, at a price vector p with $p_j < 0$, $G_{b,j}(p)$ is always positive or zero, so it is impossible for any tatonnement trajectory satisfying (D.3) to enter the region $p_j < 0$. Hence, all prices remain positive or zero, i.e. $p^t \in \mathbb{R}^m_+$ for all t. In $\mathbb{R}^m_+$, (23) is identical to (D.3), so we are done.

Proof (of Lemma 7.3.). By Lemma D.3, $G_b(p)$ is convex, compact and upper semi-continuous at p in the interior of $\Omega(p^0)$. Now, by Theorem 7.2, (D.3) has an absolutely continuous solution with $p(0) = p^0$ for some time interval $[0, \bar{t}]$, where $\bar{t} = \frac{\rho}{b} > 0$. And by Lemma D.4, this is also a solution to (20).

D.2.4. Global Existence: Proof of Lemma 7.4

Proof (of Lemma 7.4). We will prove the result for differential inclusion (D.3) and then the result follows from Lemma D.4.

We begin by defining some new notation. For any bounded demand price vector p, let $B(p, \delta)$ denote the ball around p with radius δ, let $\bar{b}(p, \delta) := \sup_{p' \in B(p,\delta)} \max_j |\nabla_j \phi(p')/h''(p'_j)|$, and let $\kappa(p) := \sup_\delta \frac{\delta}{\bar{b}(p,\delta)}$. If $\kappa(p) > 0$, then there exists at least one δ' with $\frac{\delta'}{\bar{b}(p,\delta')} \geq \frac{\kappa(p)}{2}$; for any such δ', we call $B(p, \delta')$ a *good ball* for p.

The controllability assumption provides a ball $B(p^0, \delta)$ for some $\delta > 0$, such that $\bar{b}(p^0, \delta) \leq b(p^0, 0)$. This shows $\kappa(p^0) > 0$. Using a good ball for p^0, we apply Lemma 7.3 to get a solution for some time interval $[0, t']$ with $t' > 0$. Once again, due to the assumption of controllability, there is a good ball for $p^{t'}$, to which we apply Lemma 7.3 to get a solution for some time interval $[t', t'']$ with $t'' > t'$. We repeatedly extend the trajectory in this way by additional applications of Lemma 7.3.

We claim that the process described in the last paragraph yields a trajectory over $[0, +\infty)$. Our proof is by contradiction. Suppose the contrary, i.e. the process yields a trajectory ending at but possibly not reaching some finite time $\bar{t}$.

By the controllability assumption, for any $t \in [0, \bar{t})$, all prices in p^t are bounded by $\lim_{t \nearrow \bar{t}} c(p^0, t)$, which is finite; consequently the sequence $\{p^t\}_{0 \leq t < \bar{t}}$ has a cluster point $\tilde{p}$. By the controllability assumption again, all $\frac{dp^t_j}{dt}$ are bounded by $\lim_{t \nearrow \bar{t}} b(p^0, t)$, which is again finite. Hence, the sequence $\{p^t\}_{0 \leq t < \bar{t}}$ has at most one cluster point, i.e. $\tilde{p}$ is the unique cluster point of the sequence $\{p^t\}_{0 \leq t < \bar{t}}$. Setting $p^t(t = \bar{t}) = \tilde{p}$ extends the solution to $t = \bar{t}$.

Now we have a solution over $[0, \bar{t}]$. By the controllability assumption, there exists a neighborhood of the trajectory such that each q in the neighborhood has a finite $\nabla \phi(q)/h''(q)$, bounded by $b(p^0, \bar{t})$. Let ρ

denote the minimum distance between any point in the trajectory and the boundary of the neighborhood; ρ is strictly positive. It is then easy to see that for any good ball $B(p^{t'}, \delta')$ used in the solution extension process, $\frac{\delta'}{b(p^{t'},\delta')} \geq \frac{\rho}{2b(p^0,t)}$. By Lemma 7.3, every step extends the time span of the trajectory by at least $\frac{\rho}{2b(p^0,t)}$. So after finitely many steps, the process reaches or passes beyond time $\bar{t}$, contradicting the assumption that it takes infinitely many iterations of the process to reach $\bar{t}$.

D.3. Convergence of the Solution to an Equilibrium

Claim D.3. *For any Arrow-Debreu economy in which ϕ exists, for any positive real number c, $\phi(p) = \phi(cp)$.*

PROOF. By Walras' law, $p \cdot \nabla\phi(p) = 0$. By the definition of ϕ, $\nabla\phi(p) = \nabla\phi(cp)$. By the definition of subgradient,

$$\phi(p) \geq \phi(cp) + (p - cp) \cdot \nabla\phi(cp) = \phi(cp) + (1-c)p \cdot \nabla\phi(p) = \phi(cp)$$

and

$$\phi(cp) \geq \phi(p) + (cp - p) \cdot \nabla\phi(p) = \phi(p) + (c-1)p \cdot \nabla\phi(p) = \phi(p).$$

These two inequalities imply that $\phi(p) = \phi(cp)$.

PROOF (OF LEMMA 7.5 FOR ARROW-DEBREU ECONOMIES). Let $S = \sum_\ell p_\ell$. Then $\hat{p}_j = p_j/S$.

$$\begin{aligned}
\frac{d}{dt} d_h(\hat{p}^*, \hat{p}) &= \sum_j \frac{\partial d_h(\hat{p}_j^*, \hat{p}_j)}{\partial \hat{p}_j} \cdot \frac{\partial \hat{p}_j}{\partial t} = \sum_j \frac{\partial d_h(\hat{p}_j^*, \hat{p}_j)}{\partial \hat{p}_j} \sum_k \frac{\partial \hat{p}_j}{\partial p_k} \cdot \frac{\partial p_k}{\partial t} \\
&= \sum_j \frac{\partial d_h(\hat{p}_j^*, \hat{p}_j)}{\partial \hat{p}_j} \left[\frac{1}{S} \frac{\partial p_j}{\partial t} + \sum_k \frac{-p_j}{S^2} \frac{\partial p_k}{\partial t} \right] \\
&= \frac{1}{S^2} \sum_j h''(\hat{p}_j) \cdot (\hat{p}_j^* - \hat{p}_j) \left[S \frac{\nabla_j \phi(p)}{h''(p_j)} - p_j \sum_k \frac{\nabla_k \phi(p)}{h''(p_k)} \right] \\
&= \frac{1}{S} \sum_j \frac{h''(\hat{p}_j)}{h''(p_j)} \nabla_j \phi(p) \cdot (\hat{p}_j^* - \hat{p}_j) - \frac{1}{S^2} \left(\sum_k \frac{\nabla_k \phi(p)}{h''(p_k)} \right) \sum_j p_j h''(\hat{p}_j) \cdot (\hat{p}_j^* - \hat{p}_j).
\end{aligned}$$

When h is the kernel of the KL-divergence, $h''(\hat{p}_j) = \frac{1}{\hat{p}_j} = \frac{S}{p_j}$. Thus $p_j h''(\hat{p}_j) = S$ and $\frac{h''(\hat{p}_j)}{h''(p_j)} = S$. It follows that

$$\frac{d}{dt} d_h(\hat{p}^*, \hat{p}) = \sum_j \nabla_j \phi(p) \cdot (\hat{p}_j^* - \hat{p}_j) - \frac{1}{S} \left(\sum_k \frac{\nabla_k \phi(p)}{h''(p_k)} \right) \left(\sum_j (\hat{p}_j^* - \hat{p}_j) \right).$$

Since $\hat{p}^*$ and $\hat{p}$ are both normalized prices, the second term on the RHS is zero. Noting that $\nabla_j \phi(p) = \nabla_j \phi(\hat{p})$, and by Lemma D.3, we see that the rest of the argument is the same as for the Fisher economies.