Waterlevel Algorithm (for Integral Bipartite Matching)

Match j to $\operatorname{argmax}_{i:i\sim j}\{y_i\}$, $\quad y_i = \frac{\sum_j x_{ij}}{B_i}$.

Charging:

Increment $\alpha_i B_i$ by $g(y_i)$.

Set $\beta_j = 1 - g(y_i)$.

Analysis:

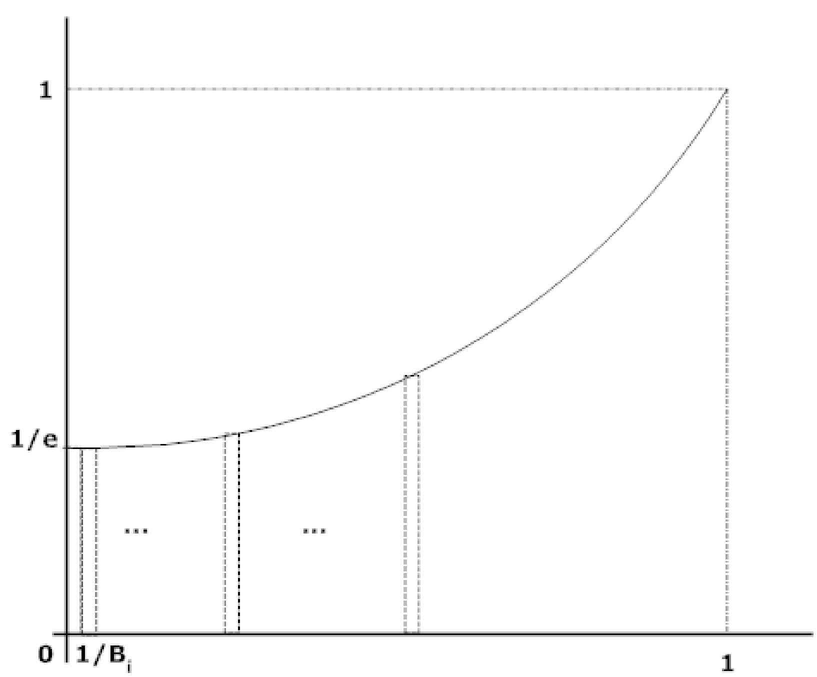
$\alpha_i B_i = g(0) + g(\frac{1}{B_i}) + ... + g(y_i^f)$

$\alpha_i = g(0)\frac{1}{B_i} + g(\frac{1}{B_i})\frac{1}{B_i} + ... + g(y_i^f\frac{1}{B_i})$

as $B_i \to \infty \quad \alpha_i = G(y_i^f) - G(0)$

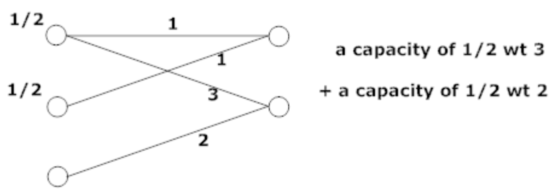
$$\frac{e^{\frac{i+1}{B_i}}}{e^{\frac{i}{B_i}}} = e^{\frac{1}{B_i}}$$

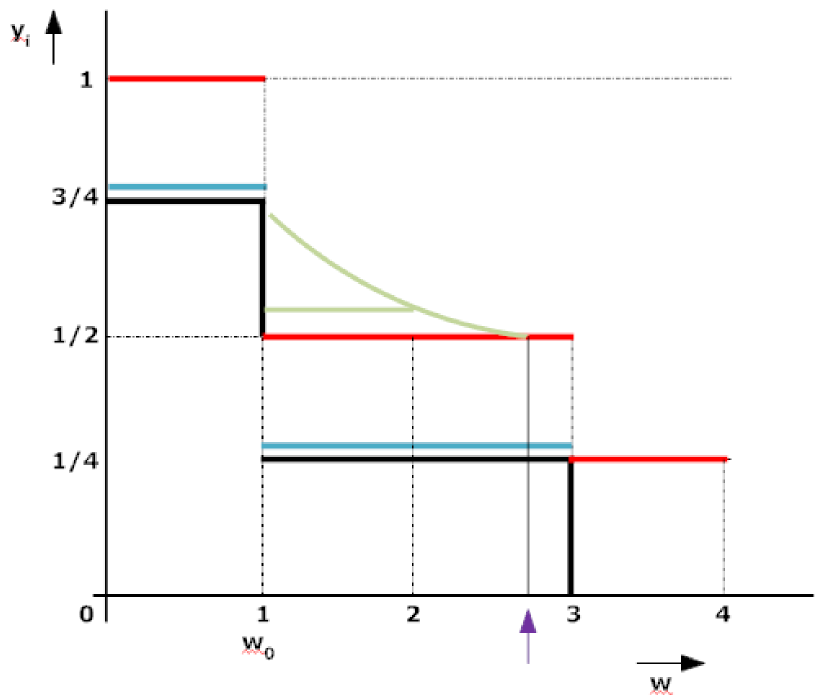
$\beta_j \geq 1 - g(y_i^f)$



Free-disposable Problem

- Edges can be discarded.
- Vertices in L can be rematched.
- Vertices in R cannot be rematched.





$y_0 : \mathbb{R}_+ \longrightarrow [0, 1]$

$y_i(w) = \sum_{j : w_i j \geq w} x_{ij}$

i gets $\int_{w_0}^{w} g(y_i(z))dz$, j gets $\int_{w_0}^{w} g(y_i(z))dz$

$y_i(w_0)$: We haven't matched any edge of $wt < w_0$.

Otherwise: $y_i(w_0 + \epsilon) = 1$ for some $\epsilon > 0$.

$w_0 = max\{w : y_i(w) = 1\}$

Algorithm:

When j arrives:

Repeat:

- match dx of j to $\underset{i}{\operatorname{argmax}} \quad \int_0^{w_{ij}} (1 - g(y_i(z)))$

Until:

- either $\sum_i x_{ij} = 1$ or $\underset{i}{\operatorname{argmax}} \quad \int_0^{w_{ij}} (1 - g(y_i(z))) = 0$

Charging:

If dx of i is matched to j:

Increment α_i by $dx. \int_{w_{i0}}^{w_{ij}} g(y_i(z))dz.$

Increment β_j by $dx. \int_0^{w_{ij}} (1 - g(y_i(z)))$

Analysis:

We want to show that $\forall i, j \quad \alpha_i + \beta_j \geq w_{ij}.\gamma$

$\alpha_i = \int_0^{\infty} [G(y_i^f(z)) - G(0)]dz \geq \int_0^{w_{ij}} [G(y_i^f(z)) - G(0)]$

$\beta_j \geq \int_0^{w_{ij}} [1 - g(y_i^f(z))]$

$\alpha_i + \beta_j \geq \int_0^{w_{ij}} \gamma dz + \gamma w_{ij}$

Exercise: Compare to Feldman et. al.

Primal LP:

$\max \sum_{i,j} w_{ij} x_{ij}$ s.t.

$\forall i \quad \sum_j x_{ij} \leq 1$

$\forall j \quad \sum_i x_{ij} \leq 1$

$x_{ij} \geq 0$

$\min \sum_i \alpha_i + \sum_j \beta_j$

$\forall i, j \quad \alpha_i + \beta_j \geq w_{ij}$

Random Order Model

- Vertices in j arrive in a random order.
- E[ALG] is over the random order of the input.
- OPT does not depend on the order.

Adwords or BA problem $b_{ij} \ll B_i$ (Integral)

Greedy Algorithm:

When j arrives, match it to $\underset{i}{\text{argmax}}\{b_{ij} : y_i \leq 1 - \frac{b_{ij}}{B_i}\}$

$y_i = \frac{\sum_j b_{ij} x_{ij}}{B_i}$

Charging:

Pick Z_j from [0, 1] u.a.γ $\forall j \in R$

js arrive in increasing order of $Z_j \Rightarrow j$ arrives at "time" Z_j.

If i is matched to j, then;

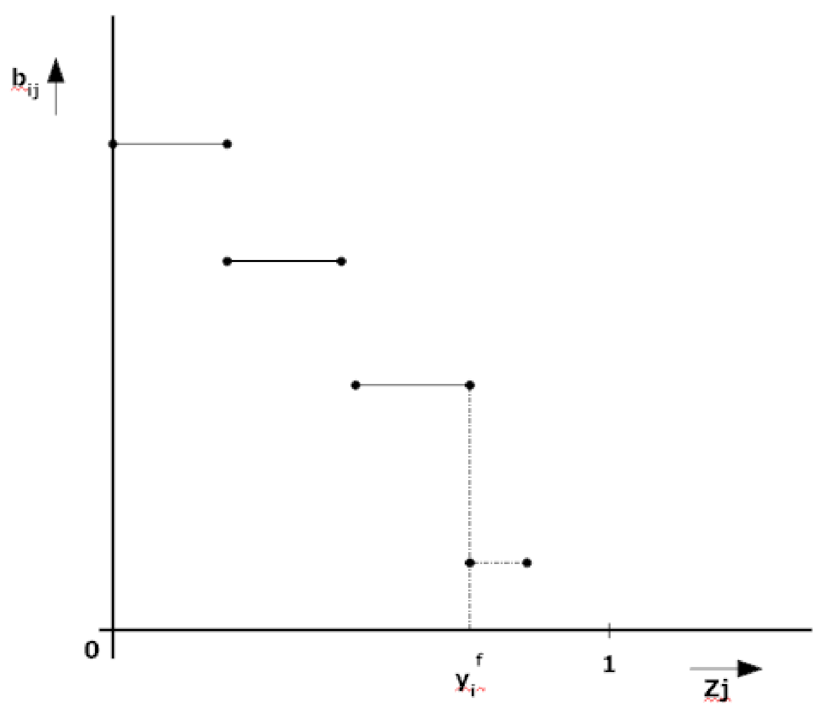
- Increment $\alpha_i B_i$ by $(1 - g(Z_j)).b_{ij}$

- Set $\beta_j = g(Z_j) b_{ij}$

Analysis:

We want to prove that $\forall i, j \quad \alpha_i b_{ij} + \beta_j \geq \gamma b_{ij}$.

y_i^f = time at which i exhausts its budget in the absence of j.



$\forall Z_j \in [0, y_i^f] \quad \beta_j \geq g(Z_j) b_i j$

$E_{Z_j}[\beta_j] \geq \int_0^{y_i^f} g(Z).b_{ij} dz = [G(y_i^f) - G(0)] b_{ij}$

Claim: $\alpha_i \geq (1 - g(y_i^f))$

Proof: $\forall j\ x_{ij} > 0, Z_j \leq y_i^f$

$\alpha_i\beta_j = \sum_j b_{ij}x_{ij}(1 - g(Z_j)) \geq \sum_j b_{ij}x_{ij}(1 - g(y_i^f)) = (B_i - b_{ij})(1 - g(y_i^f))$

$\Rightarrow \alpha_i \geq (1 - g(y_i^f))(1 - \frac{b_{ij}}{B_i})$