JAN 11 NOTES FROM CSE 599: ONLINE ALGORITHMS

Previously, we saw that the competitive ratio for algorithms on the fractional online bipartite matching problem is bounded above by $1 - \frac{1}{e}$. Below we will see an algorithm that achieves this bound.

1. Water Level Algorithm

First, we describe the algorithm. Let x_{ij} denote the value placed on edge (i, j) (the amount of water flowing from j to i). Let $y_i = \sum_{j \sim i} x_{ij}$ be the sum of the values of all the edges incident to i (the level of water in bin i).

Now, when a new vertex j arrives, we do the following: While $\sum_{i \sim j} x_{ij} < 1$ and $\min_{i' \sim j} y_{i'} < 1$, increment x_{ij} by dx for each i in $\text{argmin}_{i' \sim j} y_{i'}$.

Proposition 1. *The algorithm described above has competitive ratio at least* $1 - \frac{1}{e}$.

Proof. Let α_i denote the total money placed on a left vertex i by the algorithm (ALG) and let β_j denote the total money placed on a right vertex j by ALG. Define the function $g(x) = e^{x-1}$. Let y_i^f denote the value of y_i at the termination of the algorithm (similarly for α_i^f,...).

Fix an edge (i, j). If the algorithm assigns value dx to the edge (i, j), then we increment α_i by $g(y_i)dx$ and β_j by $(1 - g(y_i))dx$.

Now if we show that $\alpha_i^f + \beta_i^f \geq 1 - \frac{1}{e}$ for edge (i, j), the proof will be complete. Note that

$$\alpha_i^f = \int_0^{y_i^f} g(y_i) dy_i = g(y_i^f) - \frac{1}{e}$$

By the monotonicity of g, we have that $g(y_i) \leq g(y_i^f)$ throughout the process. Hence $\beta_j^f \geq 1 - g(y_i^f)$. Thus we have that $\alpha_i^f + \beta_i^f \geq 1 - \frac{1}{e}$. $\square$

Remark 2. The g in the above proof can be derived from the properties we desire it to have. If we let $G(y) = \int_0^y g(x)dx$ and γ be the target competitive ratio, then we have the desired properties $G(1) - G(0) = \gamma$ and $G(y_i^f) - G(0) + 1 - g(y_i^f) = \gamma$. From these equations, we can get a differential equation (by differentiating the second equation) and boundary conditions. Solving these, we obtain the function g.

We can also look at this from the viewpoint of linear programming (LP). Written as an LP, the fractional bipartite matching problem has the form:

$\max \sum_{(i,j) \in E} x_{ij}$ such that

- $\sum_{j \sim i} x_{ij} \leq 1$ for every $i \in L$
- $\sum_{i \sim j} x_{ij} \leq 1$ for every $j \in R$
- $x_{ij} \geq 0$

This LP has the following dual:

$\min \sum_i \alpha_i + \sum_j \beta_j$ such that

- $\alpha_i + \beta_j \geq 1$ for all $(i, j) \in E$
- $\alpha_i,\ \beta_j \geq 0$

Above, we showed that $\alpha_i + \beta_j \geq \gamma = 1 - \frac{1}{e}$. Thus, if we scale the constraints in the dual problem by γ (i.e. we require $\geq \gamma$ instead of ≥ 1), then we obtain a feasible dual problem. This scaling corresponds to multiplying the objective function in the primal problem by γ. Thus, using weak duality we have the following inequality:

$$\gamma\text{PRIMAL} \leq \gamma\text{OPT} \leq \text{FEASIBLE DUAL}$$

Now by noting that our algorithm determines an instance of FEASIBLE DUAL and by dividing by OPT, we see a proof that the competitive ratio is bounded below by γ.

2. Budget Allocation (Adwords) Problem

For this problem, we introduce the following notation. Let L (the left vertices) be a set of advertisers. Each $i \in L$ has a budget B_i. Let R (the right vertices) be a set of "queries." For every $j \in R$ and $i \in L$, we have a bid b_{ij}. Now each query can be matched to at most one advertiser. Then the quantity that we want to maximize is:

$$\sum_i \min \left\{ \sum_{j:j \to i} b_{ij},\, B_i \right\}$$

For the adwords case of this problem, we make the additional assumption that $\frac{b_{ij}}{B_i} \ll 1$.

The fractional version of this problem is easier to work with and, under the adwords assumption, produces similar results. In the fractional setting we define x_{ij} to be the weight that we place on an edge b_{ij}. Then the problem becomes the following LP:

max $\sum_{(i,j)} b_{ij} x_{ij}$ such that

- $\sum_i x_{ij} \leq 1$ for every $j \in R$
- $\sum_j b_{ij} x_{ij} \leq B_i$ for every $i \in L$
- $x_{ij} \geq 0$

This problem has the dual:

min $\sum_i \alpha_i B_i + \sum_j \beta_j$ such that

- $\alpha_i b_{ij} + \beta_j \geq b_{ij}$ for all (i, j)
- $\alpha_i,\, \beta_j \geq 0$