Prophet Inequality

Note Title 2/21/2013

- $v_1, v_2, \ldots, v_i, \ldots v_n$ arrive online.
- when v_i arrives, pick v_i or not. (irrevocable)
- only pick one.

Cardinal version of secretary problem.
Assume $v_i \sim F_i$, probability distribution.
Algo. knows $F_i, \forall i$.

$E[OPT] = \mu = E[\max_i \{v_i\}]$. $CR = E[ALG]/\mu$.

Algo :- Pick the first v_i s.t. $v_i \geq \frac{\mu}{2}$.

Theorem :- $CR = 1/2$. i.e, $E[ALG] \geq \mu/2$.

Proof :- Let $Y = ALG$, $Z = OPT$, $T = \mu/2$, threshold

$$Y = \begin{cases} Y - T + T & \text{if } Y \geq T \\ 0 & \text{if } Y \leq T \end{cases}$$

$$(Y-T)_+ = \begin{cases} Y-T & \text{if } Y \geq T \\ 0 & \text{o.w.} \end{cases}$$

$$\min\{Y,T\} = \begin{cases} T & \text{if } Y \geq T \\ 0 & \text{o.w} \end{cases}$$

$$Y = (Y-T)_+ + \min\{Y,T\}$$

$$p = \Pr[Y \geq T].$$

$$E[Y] = E[Y-T \mid Y \geq T] \cdot p + p \cdot T \geq p \cdot T$$

$$z = \begin{cases} z & \text{if } z \le T \\ T + y - T & \text{if } z = y \\ T + y - T + z - y & \text{if } z > y \end{cases}$$

$$\therefore z = \min\{z, T\} + (y - T)_+ + (z - y)_+$$

$$E[\min\{z, T\}] \le T$$

$$\Pr[z > y] \le p^2.$$

$$\mu \le (1-p)T + E[y]$$

$$E[y] \ge (1-p)\cdot T + p\cdot(\mu - T).$$