FEBRUARY 1 NOTES FROM CSE 599: ONLINE ALGORITHMS

We continue with the i.i.d. model with unknown distributions.

1. Budgeted Allocation Problem

Last time, we worked on Bipartite Matching. A suggested exercise was to extend the technique to the Budgeted Allocation problem. We now discuss how this works.

First, we change the problem from Bipartite Matching to Budgeted Allocation:

- affine side L, where we have a budget B_i for each $i \in L$ (each advertiser)
- There's a probability distribution with support $\hat{R}$
  - $j \in \hat{R}$ (query) is a "vertex" identified by its neighbors in L with bids b_{ij}
  - $p(j)$ is the probability of j
  - $\sum_{j \in \hat{R}} p(j) = 1$

Continuing, the LP for $\overline{\text{OPT}}$ (defined on the "expected instance") changes slightly to:

$$\begin{aligned} \overline{\text{OPT}} = \max \quad & \sum_{i,j} b_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_j b_{ij} x_{ij} \le B_i, \quad \forall i \\ & \sum_i x_{ij} \le m p_j, \quad \forall j \\ & x_{ij} \ge 0 \end{aligned}$$

The lemma from before ($\overline{\text{OPT}} \ge \mathbb{E}[\text{OPT}]$) still holds, and the proof is only slightly altered.

Proof Sketch. X_{ij} = number of times a copy of j is matched to i
$x_{ij} = \mathbb{E}[X_{ij}]$

$$\forall i, \sum_j b_{ij} X_{ij} \le B_i \quad \Rightarrow \quad \sum_j b_j x_{ij} \le B_i$$

etc. ...

$$\begin{aligned} \sum_{ij} b_{ij} x_{ij} &= \mathbb{E}\left[\sum b_{ij} X_{ij}\right] \\ &= \mathbb{E}[\text{OPT}] \le \overline{\text{OPT}} \end{aligned}$$

□

The theorem (greeedy is $1 - 1/e$ competitive) still holds, but the details change.

2. Resource Allocation Problem

We return to the Resource Allocation Problem and give a new algorithm.

- Resources: $i \in L$, capacity C_i, $|L| = n$
- Requests: j arrive online with set of feasible options given by $\mathcal{F}_j$
- Options: for each option $k \in \mathcal{F}_j$, $\forall i$, consumption is $a(i,j,k)$ and profit is $w(j,k)$

$$\begin{aligned} \max \quad & \sum_{j,k} w_{jk} x_{jk} \\ \text{s.t.} \quad & \sum_{k} x_{jk} \le 1, \quad \forall j \\ \sum_{j,k} a(i,j,k) x_{jk} & \le C_i, \quad \forall i \\ x_{jk} & \ge 0 \end{aligned}$$

Similarly, for the i.i.d. model, we have a distribution on j's (requests), say with support $\hat{R}$. For simplicity, assume as before that $p(j) = \frac{1}{m}\ \forall j \in \hat{R}$.

This time,
Goal: Design algorithm s.t. $\forall \epsilon, \delta > 0$, w.p. $1-\delta$, ALG $\ge (1-\epsilon)$OPT.
Assume:

- know m
- (for simplicity) know $\overline{\text{OPT}}$
- $\forall i, j, k, \frac{C_i}{a(i,j,k)} \ge \kappa \ge \frac{c \lg(\frac{n}{\delta})}{\epsilon^2}$ for some universal constant c
- $\frac{\overline{\text{OPT}}}{w(j,k)} \ge \kappa$

First, consider Pure Random Algorithm

- knows $\hat{R}$
- knows optimal solution to LP, x
- non-adaptive (makes decisions ahead of time)
- will satisfy goal

Algorithm:
Given j, pick option k w.p. $\frac{x_{jk}}{1+\epsilon}$ (a scaling of the LP solution)
X_i = total consumption of resource i
$= \sum_{t=1}^{m} X_{it}$ (they are independent)
$\forall i, \forall t, \mathbb{E}[X_{it}] = \sum_j \frac{1}{m} \sum_k \frac{x_{jk}}{1+\epsilon} a(i,j,k) \le \frac{C_i}{m(1+\epsilon)}$
$\mathbb{E}[X_i] \le \frac{C_i}{1+\epsilon}$
want to conclude that $\Pr[X_i \ge C_i] \le \frac{\delta}{2n}$(with high probability, we are not over capacity)
for simplicity,

$$\begin{aligned} a(i,j,k) &\in [0,1] \qquad & C_i &\ge \kappa \\ w(j,k) &\in [0,1] \qquad & \overline{\text{OPT}} &\ge \kappa \end{aligned}$$

This follows from Chernoff bounds, but we prove them.

2.1. **Chernoff Bounds.**

$$\Pr[X_i \geq C_i] = \Pr[(1+\epsilon)^{X_i} \geq (1+\epsilon)^{C_i}]$$
$$\leq \frac{\mathbb{E}[(1+\epsilon)^{X_i}]}{(1+\epsilon)^{C_i}}$$

The inequality is due to Markov's Inequality: $\Pr[X \geq a] \leq \frac{\mathbb{E}[x]}{a}$

$$\mathbb{E}\left[(1+\epsilon)^{X_i}\right] = \mathbb{E}\left[(1+\epsilon)^{\sum_{t=1}^{m} X_{it}}\right] \tag{1}$$
$$= \mathbb{E}\left[\prod_t (1+\epsilon)^{X_{it}}\right] \tag{2}$$
$$= \prod_{t=1}^{m} \mathbb{E}\left[(1+\epsilon)^{X_{it}}\right] \tag{3}$$
$$\leq \prod_{t=1}^{m} \mathbb{E}\left[1+\epsilon X_{it}\right] \tag{4}$$
$$\leq \prod_{t=1}^{m} \left(1 + \frac{\epsilon C_i}{m(1+\epsilon)}\right) \tag{5}$$
$$\leq \prod_{t=1}^{m} e^{\frac{\epsilon C_i}{m(1+\epsilon)}} \tag{6}$$
$$= e^{\frac{\epsilon C_i}{1+\epsilon}} \tag{7}$$

Line 3 is because of the independence of the $X_{it}s$. Line 4 is because $(1+\epsilon)^x \leq (1+\epsilon x)\, \forall x \in [0,1]$. Line 6 is because $1 + x \leq e^x$, based on the Taylor series of e^x.

$$\Pr[X_i \geq C_i] \leq \frac{e^{\frac{\epsilon C_i}{1+\epsilon}}}{(1+\epsilon)^{C_i}} \tag{8}$$
$$= \left[\frac{e^{\epsilon}}{(1+\epsilon)^{1+\epsilon}}\right]^{\frac{C_i}{1+\epsilon}} \tag{9}$$
$$\leq e^{\frac{-\epsilon^2 C_i}{2(1+\epsilon)}} \tag{10}$$
$$\leq e^{\frac{-\epsilon^2 C_i}{4}} \tag{11}$$
$$\leq e^{-\log(\frac{2n}{\delta})} \tag{12}$$
$$= \frac{\delta}{2n} \tag{13}$$

Line 10 is because $\frac{e^x}{(1+x)^{1+x}} \leq e^{\frac{-x^2}{2}}$. Line 12 is because $\frac{\epsilon^2 C_i}{4} \geq \log \frac{2n}{\delta}$ (see inequality in assumptions).

2.2. **Back to Pure Random.** Let W be the profit of Pure Random, $\sum_t W_t$.

$$\mathbb{E}[W_t] = \sum_j \frac{1}{m} \sum_k \frac{X_{jk}}{1+\epsilon} w(j,k) = \frac{\overline{\text{OPT}}}{m(1+\epsilon)}$$

$$\mathbb{E}[W] = \frac{\overline{\mathrm{OPT}}}{1+\epsilon}$$

$$\Pr\left[W \leq \frac{\overline{\mathrm{OPT}}}{1+\epsilon}(1-\epsilon)\right] \leq \frac{\delta}{2n}$$

Therefore, w.p. $1-\delta$,

- no capacity is exceeded
- $w \geq \frac{\mathrm{OPT}(1-\epsilon)}{1+\epsilon} \geq \mathrm{OPT}(1-2\epsilon)$

Proof.

$$\begin{aligned}\Pr\left[W \leq \frac{\overline{\mathrm{OPT}}}{1+\epsilon}(1-\epsilon)\right] &= \Pr[(1-\epsilon)^w \geq (1-\epsilon)^{\cdots}] \\ &\leq \frac{\mathbb{E}[(1-\epsilon)^W]}{(1-\epsilon)^{\cdots}}\end{aligned}$$

$$\begin{aligned}\mathbb{E}[(1-\epsilon)^W] &= \mathbb{E}[(1-\epsilon)^{\sum_t W_t}] \\ &= \prod_t \mathbb{E}[(1-\epsilon)^{W_t}] \\ &\leq \prod_t \mathbb{E}[1-\epsilon^{W_t}] = \prod_t \left(1-\epsilon\frac{\overline{\mathrm{OPT}}(1-\epsilon)}{m(1+\epsilon)}\right) \\ &\leq \prod_{t=1}^m e^{\frac{-\epsilon\ldots}{m}} = e^{\frac{-\epsilon\overline{\mathrm{OPT}}(1-\epsilon)}{(1+\epsilon)}}\end{aligned}$$

$$\Pr\left[W \leq \frac{\mathrm{OPT}(1-\epsilon)}{1+\epsilon}\right] \leq \frac{e^{\frac{-\epsilon\overline{\mathrm{OPT}}(1-\epsilon)}{1+\epsilon}}}{(1-\epsilon)^{\frac{\mathrm{OPT}(1-\epsilon)}{1+\epsilon}}} \cdots \leq \frac{\delta}{2n}$$

The last dots are where we should do a similar approximation. □

2.3. **An Alternative.**

$$\begin{aligned}\Pr[X_i \geq C_i] &\leq \Pr[(1+\epsilon)^{X_i} \geq (1+\epsilon)^{C_i}] \\ &= \Pr[(1+\epsilon)^{\frac{KX_i}{C_i}} \geq (1+\epsilon)^K] \\ &\leq \frac{\mathbb{E}[(1+\epsilon)^{\frac{KX_i}{C_i}}]}{(1+\epsilon)^K} \leq \frac{\delta}{2n}\end{aligned}$$

Suppose we showed

$$\sum_i \mathbb{E}[(1+\epsilon)^{\frac{KX_i}{C_i}}] \le (1+\epsilon)^K \frac{\delta}{2}$$

$$\mathbb{E}[\max_i (1+\epsilon)^{\frac{KX_i}{C_i}}] \le \mathbb{E}[\sum_i (1+\epsilon)^{\frac{KX_i}{C_i}}]$$

$$\le (1+\epsilon)^K \frac{\delta}{2}$$

$$\Pr[\max_i \frac{KX_i}{C_i} \ge K] \le \frac{\delta}{2}$$

$$\Pr[\max_i \frac{X_i}{C_i} \ge 1] \le \frac{\delta}{2}$$

Equivalently, Pure Random can be thought of as minimizing

$$\Phi = \frac{\sum_i (1+\epsilon)^{\frac{KX_i}{C_i}}}{(1+\epsilon)^K} + \frac{(1-\epsilon)^{\frac{KW}{\text{OPT}}}}{(1+\epsilon)^K}$$

$$\mathbb{E}[\Phi^P] \le \delta$$

We will show that $\mathbb{E}[\Phi^t] \le \delta$.

$$\mathcal{H}^t = A_1 \ldots A_t P_{t+1} \ldots P_m$$

$$\mathbb{E}[\phi^{\mathcal{H}^t} | A_1 \ldots A_{t-1}] \le \mathbb{E}[\phi^{\mathcal{H}^{t-1}} | A_1 \ldots A_{t-1}]$$