Tatonnement Beyond Gross Substitutes? Gradient Descent to the Rescue *

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ABSTRACT

Tatonnement is a simple and natural rule for updating prices in Exchange (Arrow-Debreu) markets. In this paper we define a class of markets for which tatonnement is equivalent to gradient descent. This is the class of markets for which there is a convex potential function whose gradient is always equal to the negative of the excess demand and we call it Convex Potential Function (CPF) markets. We show the following results.

- CPF markets contain the class of Eisenberg Gale (EG) markets, defined previously by Jain and Vazirani.
- The subclass of CPF markets for which the demand is a differentiable function contains exactly those markets whose demand function has a symmetric negative semi-definite Jacobian.
- We define a family of continuous versions of tatonnement based on gradient descent using a Bregman divergence. As we show, all processes in this family converge to an equilibrium for any CPF market. This is analogous to the classic result for markets satisfying the Weak Gross Substitutes property.
- A discrete version of tatonnement converges toward the equilibrium for the following markets of complementary goods; its convergence rate for these settings is analyzed using a common potential function.
  - Fisher markets in which all buyers have Leontief utilities. The tatonnement process reduces the distance to the equilibrium, as measured by the potential function, to an ϵ fraction of its initial value in $O(1/\epsilon)$ rounds of price updates.
  - Fisher markets in which all buyers have complementary CES utilities. Here, the distance to the equilibrium is reduced to an ϵ fraction of its initial value in $O(\log(1/\epsilon))$ rounds of price updates.

This shows that tatonnement converges for the entire range of Fisher markets when buyers have *complementary* CES utilities, in contrast to prior work, which could analyze only the *substitutes* range, together with a small portion of the complementary range.

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1. INTRODUCTION

One of the central questions in economics regarding equilibrium models such as the Walrasian model of a market is, "do markets reach an equilibrium", and if so, "how". In particular the question, when does the *tatonnement* process converge to an equilibrium, has been widely studied [1, 18, 16, 5, 7]. The tatonnement process is a natural, simple and distributed family of price adjustment rules. It is broadly defined in terms of the following criteria: if the demand for a good is more than the supply, increase the price of the good and vice versa, decrease the price when the demand is less than the supply. The price adjustment for each good is in the direction of its own excess demand and is independent of the demand for other goods. Classically, tatonnement has been thought of as a *continuous* process, with price adjustments and demand responses happening continuously and instantaneously. A computer science approach is to consider updates at *discrete* time intervals and to bound the number of updates required (though discrete updates were also considered in the economics literature as early as the 60s [16]).

What is known to date is essentially that the tatonnement process converges whenever the market satisfies the *(weak) gross substitutes (WGS)* property. A market satisfies the weak gross substitutes property if increasing the price of one good does not decrease the demand for any other good.[1] The

[1] The (strict) gross substitutes property says that increasing

seminal paper of [1] showed this for a continuous version of the tatonnement process. In recent years a discrete version of the process has been shown to converge quite fast as well, by [7]. This paper is motivated by the quest for other broad classes of markets for which the same holds true. Of particular interest are markets that exhibit *complementarity*, such as the *Constant Elasticity of Substitution (CES)* utilities and *Leontief* utilities. (See Section 2 for formal definitions.)

For the WGS case, all the results rely on very strong properties of WGS markets that are naturally helpful in showing that tatonnement converges. One example of such a property is that the extreme prices always move in, i.e., the bound on the ratio of the current price to the equilibrium price is guaranteed to shrink [7]. Another example is that for some markets (called *Fisher* markets) the equilibrium can be reached by starting with very small prices and *monotonically* increasing them. It is easy to show that such strong properties cease to hold in the complementary regime. Therefore new techniques are needed to handle such markets.

In this paper we relate the tatonnement process to another simple and natural process: *gradient descent*. Gradient descent is a family of algorithms used to minimize convex functions. It works by starting at some point and moving in the direction of the negative of the gradient. We consider the class of markets for which the tatonnement process is formally equivalent to performing gradient descent on a convex function. In particular, we define the class of Convex Potential Function (CPF) markets to be those markets for which there is a convex potential function whose gradient[2] is always equal to the negative of the excess demand. We show that this class contains the class of Eisenberg-Gale (EG) markets introduced by Jain and Vazirani [12].

The subclass of CPF markets for which the demand is differentiable can be characterized in terms of the Jacobian[3] of the demand function. These are exactly those markets for which the Jacobian of the demand function is always symmetric and negative semi-definite.[4] We call this class the Convex Conservative Vector Field (CCVF) markets, since functions that have a symmetric Jacobian are called conservative vector fields. The aforementioned CES and Leontief utilities along with many other interesting markets (in the Fisher market model) are contained in the intersection of EG markets and CCVF markets.

The equivalence with gradient descent opens up the entire tool box developed to analyze gradient descent and provides a principled approach to show convergence of the tatonnement process. For the entire class of CPF markets, we show that a continuous version of tatonnement converges to an equilibrium. For the special cases of CES and Leontief utilities, we show stronger convergence results by proving certain structural properties of the corresponding convex functions for these markets.

We now summarize the main contributions of the paper.

- The class of Eisenberg-Gale (EG) markets contains all Fisher markets for which the equilibrium allocation is captured by a certain type of convex program called the Eisenberg-Gale-type (EG-type) convex program. We show that EG markets are CPF markets, by explicitly constructing a convex potential function (Theorem 3.1). In fact, the potential function is essentially the dual of the corresponding EG-type convex program.

- We show that a family of continuous versions of the tatonnement process converges to the equilibrium for any CPF market. This family is derived by considering gradient descent with respect to any *Bregman divergence* and taking the limit as the step size goes to zero (Theorem 4.1). This mirrors the classic result of [1] that shows a similar result for gross substitutes markets.

- For Leontief utilities, we show a linear convergence for a discrete version of the process, that is, the number of time steps required to reduce the distance from the equilibrium to an ϵ fraction of its initial value, as measured by the potential function, is $O(1/\epsilon)$ (Theorem 5.1).[5] This follows from a general result of [3] that shows a linear convergence of gradient descent with Bregman divergences whenever the convex function satisfies a certain sandwiching property.[6] We show that the potential function in this case satisfies this sandwiching property for an appropriate choice of parameters with respect to the KL-divergence.

- For CES utilities we show a log convergence, that is, the number of time steps required to reduce the distance from the equilibrium to an ϵ fraction of its initial value, again as measured by the potential function, is $O(\log(1/\epsilon))$ (Theorem 6.4). This is obtained by showing that the potential function in this case satisfies a stronger sandwiching property. This stronger property is reminiscent of strong-convexity but to the best of our knowledge, this particular property has not been used before. We also note that when reasonably near to equilibrium, the potential function has value $\Theta(\sum_j z_j^2 p_j)$, where z_j is the excess demand for good j and p_j is its price (Lemmas 6.2 and 6.3).

Related work

The stability of the tatonnement process has been considered to be one of the most fundamental issues in general equilibrium theory. The tatonnement process was originally defined by Walras [19] in the same paper in which he defined the first equilibrium model. The textbook of Mas-Colell, Whinston and Green [15] contains a good summary of the classic results. The two most important results are that of Arrow, Block and Hurwitz [1], who showed that a continuous version of the tatonnement process converges to an equilibrium for gross substitutes markets and that of Scarf [18],

the price of one good strictly increases the demand for any other good.

[2] More generally, the potential function need not be differentiable and the demand need not be unique, in which case the equivalence is between the sub-gradient of the potential function and the set of excess demand vectors.

[3] Recall that the *Jacobian* of a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^n$ is the matrix whose (i, j) entry is the rate of change of the i^{th} component of the function with respect to a change in the j^{th} co-ordinate.

[4] By contrast, if the off-diagonal entries of the Jacobian are all positive, then the market satisfies the weak gross substitutes property.

[5] The $O()$ hides market dependent parameters.

[6] Actually we observe that a slightly weaker version of the property suffices.

who showed an example of a market where the tatonnement does not converge; in fact, it exhibits cyclic behavior.

More recently discrete versions of tatonnement have been studied. Codenotti et al. [5] consider a tatonnement-like process that required some coordination among different goods and showed polynomial time convergence for WGS markets. Cole and Fleischer [7] were the first to establish fast convergence for a truly distributed discrete version of the tatonnement, once again for a class of WGS markets. Cheung, Cole and Rastogi [4] extend this result slightly beyond WGS markets, to CES utilities for a limited range of parameters.[7] In comparison, our results cover the entire range of parameters for CES utilities. Fleischer et al. [10] also consider price dynamics that are similar to tatonnement but they also need coordination and further, the results concern the average price throughout the process rather than convergence of the sequence.

In a similar spirit to this paper, Birnbaum, Devanur and Xiao [2] considered another distributed process called the Proportional Response (PR) dynamics for the linear utilities case, showed its equivalence to gradient descent with KL-divergence for a different convex function and obtained convergence rates for the process. The PR dynamics works in the space of offers rather than the space of prices, which is why the corresponding convex function is different. For linear utilities, the PR dynamics are more appropriate than tatonnement, especially since the demand function is not continuous. [2] prove a certain convergence result (Theorem 2.1) which we use in this paper to show convergence for the case of Leontief utilities.

EG markets were defined by Jain and Vazirani [12], after observing that many markets in the Fisher model had similar convex programs that captured the equilibrium. The following is a brief list of such markets: Eisenberg and Gale [9] gave a convex program for the linear utilities case, [6] gave one for Leontief utilities and for CES utilities, [13] for homothetic utilities with production, and [14] for certain network-flow markets. [12] showed many algorithmic and structural properties of such markets.

2. PRELIMINARIES

A Walrasian market model has m divisible goods and n agents. Each agent i has a utility function $u_i : \mathbb{R}_+^m \to \mathbb{R}$ that specifies the agent's utility for a given bundle of goods. Each agent i has an initial endowment of e_{ij} amount of good j. The supply of good j, $w_j := \sum_i e_{ij}$ is the total endowment of good j among all the agents. W.l.o.g. we choose the units of measurement such that the supplies are all 1. Suppose we assign a price p_j to each good j, then the *demand* of agent i is a bundle of goods $(x_{i1}, x_{i2}, \ldots, x_{im})$ that maximizes her utility subject to the budget constraint, that she does not spend more than the value of her endowment. It is the solution to the following optimization problem:

$$\begin{aligned} &\text{maximize } u_i(x_{i1}, x_{i2}, \ldots, x_{im}) \\ &\text{s.t. } \sum_j p_j x_{ij} \le \sum_j p_j e_{ij}, \\ &\forall\, j, x_{ij} \ge 0. \end{aligned}$$

[7] CES utilities are parameterized by an exponent, ρ. When $0 < \rho \le 1$ the market is WGS, and $\rho < 0$ is when the goods are complementary. [4] analyzed the range $-1 < \rho \le 0$.

If the utility function is strictly concave, then there is a unique utility maximizing bundle when the prices are all positive, so we can talk of *the* demand of an agent. The market demand for a good j is $x_j = \sum_i x_{ij}$, the total demand for that good. This is viewed as a function of the price vector $p = (p_1, p_2, \ldots, p_m)$. A price p is an equilibrium price if the market clears, that is

$$\forall\, j, x_j = w_j = 1.$$

For notational convenience, we define the excess demand for good j as $z_j = x_j - 1$. The equilibrium condition is that every excess demand be zero. It is known that equilibrium prices exist if the utility functions are all strictly concave.

An alternate mode is the *Fisher* market model, where there is a fixed endogenous supply of each good (which is again chosen to be 1 unit). The agents have a fixed endowment of *money*, which defines their budget constraint. Let the endowment of agent i be e_i units of money. The budget constraint for agent i is $\sum_j p_j x_{ij} \le e_i$. The Fisher model is actually a special case of the exchange model.

We now define some interesting sub-classes of markets. A market satisfies the *Weak gross substitutes (WGS)* property or equivalently a market is a WGS market if increasing the price of any one good does not decrease the demand for any other good. If the demand function is continuous and differentiable, then this property can be written as

$$\frac{\partial x_j}{\partial p_{j'}} \ge 0, \quad \forall\, j \ne j'.$$

In terms of the *Jacobian* of the demand function, for a WGS market all the off-diagonal entries are non-negative.

The *Leontief* utilities are of the form $u_i = \min_j\{x_{ij}/b_{ij}\}$. One needs b_{ij} units of good j, for each good, in order to get one unit of utility. Thus Leontief utilities capture the case of *perfect complements*. It is easy to see that the demand for good j is $x_{ij} = \beta_i b_{ij}$, where $\beta_i = e_i / \sum_j b_{ij} p_j$.

Utilities with a *Constant Elasticity of Substitution (CES)* or in short, CES utilities, are of the form

$$u_i = (a_{i1} x_1^{\rho_i} + a_{i2} x_2^{\rho_i} + \cdots + a_{im} x_m^{\rho_i})^{1/\rho_i} ,$$

with $\rho_i \le 1$ and $a_{ij} \ge 0$. If $0 < \rho_i \le 1$ then the goods are substitutes; the goods are complementary when $\rho_i < 0$. Leontief utilities are obtained in the limit, as $\rho \to -\infty$. The utility function obtained in the limit, as $\rho \to 0$, is called the Cobb-Douglas utility.

An *Eisenberg-Gale-type* convex program is a convex program of the form

$$\begin{aligned} &\text{maximize } \sum_i e_i \log u_i(x_{i1}, x_{i2}, \ldots, x_{im}) \\ &\text{s.t. } \forall\, j, \sum_i x_{ij} \le 1, \ \text{(supply constraints)} \\ &\forall\, i, j, x_{ij} \ge 0. \end{aligned}$$

An *Eisenberg-Gale (EG)* market is a Fisher market for which the optimal solution and the (corresponding) Lagrange multipliers of the supply constraints in the above convex program are respectively equilibrium demand and prices for the market. Vice-versa, an equilibrium demand and prices are respectively an optimal solution and Lagrange multipliers of the supply constraints to the above convex program. Note that any strictly monotone transformation of the utility function leaves the market unchanged, since the demand function is invariant under such transformations. Thus one may need to apply suitable monotone transformations to the utility functions in order to make it into an EG market. It is known that buyers with Leontief and CES utilities in the Fisher model form EG markets.

We next present a generalized version of gradient descent and a convergence result for this version. For any strictly convex differentiable function h, the *Bregman divergence* with kernel h is defined as

$$d_h(p, q) = h(p) - h(q) - \nabla h(q) \cdot (p - q).$$

For example, the square of the Euclidean distance is obtained as a Bregman divergence, $\|p - q\|^2 = d_h(p, q)$, if $h(p) = \frac{1}{2}\|p\|^2$. Another well-known example is the KL-divergence, $\sum_j p_j \log(p_j/q_j)$ which is obtained when $h(p) = \sum_j p_j \log p_j - p_j$.

For a convex function ϕ, define the tangent hyperplane at a given point q, thought of as a linear approximation to the function, as

$$\ell_\phi(p; q) = \phi(q) + \nabla\phi(q) \cdot (p - q),$$

where $\nabla\phi(q)$ denotes an arbitrary subgradient of ϕ at q. The *generalized gradient descent* w.r.t. a Bregman divergence d_h on the convex function ϕ is a sequence $p^0, p^1, \ldots, p^t \ldots$, defined inductively (for any given starting point p^0) as

$$p^{t+1} = \arg\min_p \{\ell_\phi(p; p^t) + d_h(p, p^t)\}. \quad (1)$$

Note that if the subgradient is not unique, then this sequence need not be unique either.

For the quadratic kernel, $h(p) = \frac{1}{2}\|p\|^2$, the above update rule reduces to the usual gradient descent rule:

$$p^{t+1} = p^t - \nabla\phi(p^t).$$

If the kernel is weighted entropy, $h(p) = \sum_j \gamma_j(p_j \log p_j - p_j)$ for some weights γ_j, the update rule is

$$p_j^{t+1} = p_j^t \exp\left(\frac{-\nabla_j \phi(p^t)}{\gamma_j}\right) \quad \forall j. \quad (2)$$

Birnbaum, Devanur and Xiao [2] showed the following convergence result for gradient descent (1).

THEOREM 2.1 ([2]). *Suppose that the convex function ϕ and the kernel h are such that $\forall$ p, q,*

$$\phi(p) \le \ell_\phi(p; q) + d_h(p, q). \quad (3)$$

Let p^* be the minimizer of ϕ. Then for all t,

$$\phi(p^t) - \phi(p^*) \le \frac{d_h(p^*, p^0)}{t}.$$

We need a slightly more general version of this theorem where we require (3) to hold only for consecutive pairs p^t, p^{t+1} for all t, instead of requiring it for all pairs p, q. It is easy to see that the proof only needs this weaker condition.

The discrete version of the tatonnement process we consider will be equivalent to the gradient descent (1) where h is the weighted entropy function, i.e., the update (2) for a suitable choice of weights γ_j. The potential function ϕ will be such that $\nabla_j \phi = -z_j$. The continuous versions we consider are obtained by introducing a multiplier $1/\epsilon$ to the divergence term d_h and taking the limit as $\epsilon \to 0$. This will be presented in more detail in Section 4.

2.1 New Definitions

We now define the new classes of markets introduced in this paper. A market is said to be a *Convex Potential Function (CPF) market* if there is a convex potential function ϕ of the prices such that for all prices p, $\nabla\phi(p) = -z(p)$. By abuse of notation, we let $\nabla\phi$ denote the set of sub-gradients when ϕ is not differentiable[8] and we let $z(p)$ denote the set of excess demand vectors when the demand is not unique. The subclas of CPF markets for which the demand function is differentiable is called the *Convex Conservative Vector Field (CCVF) markets*. The following characterization of CCVF markets follows essentially immediately from Green's Theorem [11, 17].

LEMMA 2.2. *A market with a differentiable demand function is CCVF if and only if the Jacobian of its demand function is always a negative semi-definite symmetric matrix.*

PROOF. For a CCVF market, the potential function satisfies $\nabla\phi(p) = -z(p)$. As $x(p)$ and hence $z(p)$ are differentiable, it is now easy to check that the Jacobian is symmetric. Negative semi-definiteness follows because the potential function ϕ associated with the CCVF market is convex, and hence the Jacobian of $-z(p)$ is positive semi-definite.

If the Jacobian of $x(p)$ is symmetric, by Green's Theorem [11, 17], there is a function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla f = x$. Let $\phi = \sum_j p_j - f(p)$. Then $\nabla\phi(p) = \mathbb{1} - x(p) = -z(p)$. $\phi(p)$ is convex as its Jacobian is positive semi-definite, and as $\nabla\phi(p) = -z(p)$, it follows that the market is a CPF market with a differentiable demand, i.e. it is a CCVF market. $\square$

Markets with Leontief utilities and those with CES utilities are both CCVF markets. By contrast, markets with linear additive utilities are not CCVF.

3. EG MARKETS

In this section we prove the following theorem.

THEOREM 3.1. *All EG markets are CPF markets.*

The proof is by an explicit construction of a convex potential function ϕ for which $\nabla\phi(p) = -z(p)$. ϕ is actually the dual of the corresponding EG-type convex program. Recall that the EG-type convex program has variables x_{ij} for all i and j. We let X denote the set of all these variables. Also recall that the optimum solution gives the equilibrium allocation and the optimal Lagrangian multipliers of the supply constraints in the program are the equilibrium prices. The *KKT conditions* characterize the optimal solution to a convex program and the corresponding Lagrange multipliers. We now write the KKT conditions in terms of the *Lagrangian* function, which is obtained by multiplying the supply constraints by the prices and adding them to the objective function.

$$L(X, p) := \sum_i e_i \log(u_i) - \sum_{i,j} p_j x_{ij} + p \cdot \mathbb{1},$$

on the domain $\{X, p\colon \forall i, j,\ x_{ij} \ge 0;\ \forall j,\ p_j \ge 0\}$. X^* and p^* are said to satisfy the KKT conditions if

1. $X^* \in \arg\max_{X \ge 0} L(X, p^*)$ and
2. $p^* \in \arg\min_{p \ge 0} L(X^*, p)$, which is equivalent to: for all j, $p_j^* \cdot (1 - \sum_i x_{ij}^*) = 0$.

[8]We assume throughout that ϕ is continuous.

We define the potential function to be the dual objective of the EG-type convex program.

$$\phi(p) := \max_X L(X, p).$$

For common EG markets, such as the markets with linear, CES and Leontief utilities, it turns out that this function can also be written as

$$\phi(p) = \sum_j p_j - \sum_i e_i \log(\nu_i) \tag{4}$$

where ν_i is the minimum cost buyer i has to pay to obtain one unit of utility [8]. We note this for future reference.

ϕ is convex by construction. Theorem 3.1 follows by showing that the gradient of ϕ is equal to the negative of the excess demand (Lemma 3.3). However, the key property of EG markets is captured in the following lemma.

LEMMA 3.2. *For an EG market, for all p, the demand set $x(p)$ is exactly equal to* $\arg\max_X L(X, p)$.

PROOF. Suppose that the market is an EG market. Consider an equilibrium price p^*, and the equilibrium demand $x(p^*)$. Since this is an EG market, these must correspond to an optimal solution to the corresponding convex program. They must therefore satisfy the corresponding KKT conditions, which imply that $x(p^*) \in \arg\max_X L(X, p^*)$. Now note that every price p and every demand $x(p)$ form an equilibrium for some given supply; indeed simply define the supply to be equal to $x(p)$. Thus the above holds for all prices and for all demand vectors. Similarly, since any optimal solution to the convex program must also be an equilibrium, it follows from a similar argument that $\arg\max_X L(X, p^*)$ is contained in the demand set.

In fact it is easy to see that the converse is also true, that if for all p the demand is equal to $\arg\max_X L(X, p)$ then the market is an EG market. The KKT conditions are then essentially the same as the equilibrium conditions. □

LEMMA 3.3. $\nabla\phi(p) = \mathbb{1} - x(p) = -z(p)$.

PROOF. It is well known that if a convex function is defined as the maximum of many linear functions then the gradient is given by the arg max. ϕ is indeed defined as such and by Lemma 3.2 the arg max'es are given by the demands. Hence the lemma follows. □

4. CONVERGENCE OF CONTINUOUS TIME TATONNEMENT

A continuous version of the tatonnement rule is a trajectory in the price space which, to be notationally consistent with the discrete version, is denoted by p^t for all $t \in \mathbb{R}_+$. The trajectory is defined by specifying $\frac{dp}{dt}$ for all t, which we also call the "update rule". We define a family of update rules derived from the gradient descent. As before, let h be a strictly convex differentiable function. The continuous update rule w.r.t d_h is defined as follows. Let

$$p(\epsilon) := \arg\min_p \left\{\nabla\phi(p^t)\cdot p + \tfrac{1}{\epsilon} d_h(p; p^t)\right\}. \tag{5}$$

$$\tfrac{dp}{dt} := \lim_{\epsilon\to 0} \frac{p(\epsilon) - p^t}{\epsilon}. \tag{6}$$

Note that as the subgradient $\nabla\phi(p^t)$ need not be unique, $\frac{dp}{dt}$ may not be unique either. However, we can simply choose any of the legal subgradient values, as our analysis holds for every choice.

From now on, we consider the special case where h is a *separable* function, i.e., it is of the form $\sum_j h(p_j)$, for a 1-dimensional function $h : \mathbb{R} \to \mathbb{R}$. Now the minimization in (5) separates out into independent minimization problems for each good j. We will use $d_h(p_j, q_j)$ to denote $h(p_j) - h(q_j) - h'(q_j)(p_j - q_j)$, the one dimensional version of the divergence.

THEOREM 4.1. *For any separable strictly convex function h, any convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ and $p^t \in \mathbb{R}^n$ defined by (5) and (6) with any starting point p^0,*

$$\lim_{t\to\infty} p^t = p^*$$

where p^* is a minimizer of ϕ.

For any CPF market, by definition, there exists a ϕ such that $-\nabla\phi(p) = z(p)$. Substituting z for $-\nabla\phi$ in (5) and (6) gives a tatonnement update rule that, by Theorem 4.1, converges to an equilibrium for these markets.

LEMMA 4.2.

$$\frac{dp_j}{dt} = \frac{-\nabla_j\phi(p^t)}{h''(p_j^t)}, \quad \forall\ j.$$

PROOF. The minimizer in (5) must have a zero derivative:

$$\nabla_j\phi(p^t) + \tfrac{1}{\epsilon}\frac{d\big(d_h(p_j;p_j^t)\big)}{dp_j} = 0. \tag{7}$$

Since $\frac{d\left(d_h(p_j;p_j^t)\right)}{dp_j} = h'(p_j) - h'(p_j^t)$, substituting in (7) and solving for p_j gives

$$p_j(\epsilon) = h'^{-1}\left(h'(p_j^t) - \epsilon\nabla_j\phi(p^t)\right).$$

Note that since h is strictly convex, h' is strictly increasing and hence is invertible. For notational convenience, let $g(y) = h'^{-1}(y)$. Then $h'(g(y)) = y$, $h''(g(y)) \cdot g'(y) = 1$, therefore $g'(y) = \frac{1}{h''(g(y))}$. Also note that $g(h'(y)) = y$. Using these we get

$$g'(h'(y)) = \frac{1}{h''(y)}. \tag{8}$$

$$\begin{aligned}
\tfrac{dp_j}{dt} &= \lim_{\epsilon\to 0}\frac{p_j(\epsilon) - p_j^t}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{g(h'(p_j^t) - \epsilon\nabla_j\phi(p^t)) - g(h'(p_j^t))}{\epsilon}. \\
&= -g'(h'(p_j^t))\cdot\nabla_j\phi(p^t) \\
&= -\nabla_j\phi(p^t)/h''(p_j^t). \qquad \text{(by (8))}
\end{aligned}$$ □

We can use this lemma to derive the update rules for two special cases. First, when $h(p_j) = \frac{1}{2}p_j^2$, $h''(p_j) = 1$; hence, $\frac{dp_j}{dt} = -\nabla_j\phi(p)$. Next, when $h(p_j) = p_j \log p_j - p_j$, $h'(p_j) = \log p_j$ and $h''(p_j) = 1/p_j$; so, $\frac{dp_j}{dt} = -\nabla_j\phi(p)\cdot p_j$.

For the following lemma and the subsequent proof, for simplicity of exposition only, we assume that the minimizer of ϕ is unique.

LEMMA 4.3. $\sum_j \frac{d}{dt} d_h(p_j^*; p_j) < 0$, *unless $p = p^*$, where p^* is the minimizer of ϕ.*

PROOF. $d_h(p_j^*; p_j) = h(p_j^*) - h(p_j) - h'(p_j)(p_j^* - p_j)$. So,

$$\begin{aligned}\tfrac{d}{dt} d_h(p_j^*; p_j) &= -\tfrac{dh(p_j)}{dt} - \tfrac{dh'(p_j)}{dt}(p_j^* - p_j) + h'(p_j)\tfrac{dp_j}{dt} \\ &= -h''(p_j) \cdot \tfrac{dp_j}{dt} \cdot (p_j^* - p_j) \\ &\quad (\text{since } \tfrac{dh(p_j)}{dt} = h'(p_j)\tfrac{dp_j}{dt}) \\ &= \nabla_j \phi(p) \cdot (p_j^* - p_j) \text{ (from Lemma 4.2).}\end{aligned}$$

Therefore,

$$\sum_j \tfrac{d}{dt} d_h(p_j^*; p_j) = \sum_j \nabla_j \phi(p) \cdot (p_j^* - p_j) \le \phi(p^*) - \phi(p) < 0, \tag{9}$$

unless $p = p^*$. The middle inequality above follows from the convexity of ϕ: $\phi(p^*) \ge \phi(p) + \nabla\phi(p)(p^* - p)$. □

PROOF OF THEOREM 4.1. Since $d_h(p^*; p^t)$ is monotonically decreasing, and is bounded below by 0, $\lim_{t\to\infty} d_h(p^*; p^t)$ exists. Consequently, the derivative $\frac{dd_h(p^*;p^t)}{dt}$ tends to zero. By (9) this implies that $\phi(p^t)$ tends to $\phi(p^*)$ and by continuity of ϕ this implies that p^t tends to p^*. □

5. LEONTIEF UTILITIES

In this section we consider Fisher markets in which every buyer has a Leontief utility. We analyze the update rule (1) with $d_h = \gamma \cdot d_{\text{KL}}$ where d_{KL} is the KL-divergence, and γ is a market dependent parameter. This update rule amounts to

$$p_j^{t+1} = p_j^t \exp(z_j/\gamma). \tag{10}$$

We show an $O(1/\epsilon)$ convergence rate as specified in Theorem 5.1. The proof follows essentially from showing that the sandwiching property (3) required by Theorem 2.1 is satisfied, which is done in Lemma 5.2.

Here, using Equation (4), we see that ϕ is given by

$$\phi(p^t) = \sum_j p^t - \sum_i e_i \log \sum_k b_{ik} p_k^t.$$

Notation.

We let x^t denote the demands following the price update at time t, and x° denote the initial demands. We also let $\Delta p_j = p_j^{t+1} - p_j^t$ for all j.

THEOREM 5.1. *For a Leontief market, for a sequence of price updates defined by* (10)*, for all t,*

$$\phi(p^t) - \phi(p^*) \le \frac{12\gamma d_{KL}(p^*, p^0)}{t}$$

where $\gamma = 5 \cdot \max_j\{x_j^\circ + 2 \cdot \sum_i \max_k \frac{b_{ij}}{b_{ik}}\}$.

LEMMA 5.2. *If* $|\Delta p_j| \le p_j/4$*, then*

$$\phi(p^{t+1}) - \ell_\phi(p^{t+1}; p^t) \le 12\gamma d_{KL}(p^{t+1}, p^t).$$

To ensure that $|\Delta p_j| \le p_j/4$, we require that $\gamma \ge 5 \cdot \max_{j,t}\{1, x_j^t\}$, where we are maximizing the x_j^t over all the time steps of the algorithm. Of course, γ has to be picked at the beginning, at which point one may not know the value of $\max_{j,t}\{1, x_j^t\}$. In the following lemma, we show that picking $\gamma = 5 \cdot \max_j\{x_j^\circ + 2 \cdot \sum_i \max_k \frac{b_{ij}}{b_{ik}}\}$ suffices. However, if a better bound is known, that could be used instead.

LEMMA 5.3. $x_j^t \le x_j^\circ + 2 \cdot \sum_i \max_k \frac{b_{ij}}{b_{ik}}$ *for all goods j and all time t.*

PROOF. We drop the superscript t when the meaning is clear from the context. Suppose that $x_{ij} = e_i \cdot b_{ij}/\sum_k b_{ik}p_k \ge 1$; then $x_j \ge 1$ and so p_j can only increase. If $\min_l e_i \cdot b_{i\ell}/\sum_k b_{ik}p_k \ge 1$, or equivalently if $e_i/\sum_k b_{ik}p_k \ge 1/\min_l b_{i\ell}$, then every p_k for which $b_{ik} \ne 0$ can only increase, and hence the x_{ik} for which $b_{ik} \ne 0$ can only decrease; i.e. if $x_{ij} = e_i \cdot b_{ij}/\sum_k b_{ik}p_k \ge b_{ij}/\min_l b_{i\ell} = \max_k b_{ij}/b_{ik}$, x_{ij} can only decrease. Now, in one round of price changes, the prices drop by at most $\exp(\frac{1}{5})$, and hence the demands increase by at most $\exp(\frac{1}{5}) \le 2$. Thus, unless initially larger, $x_{ij} < 2 \cdot \max_k b_{ij}/b_{ik}$[9]. Thus $x_{ij} \le \max_k\{x_{ij}^\circ, 2 \cdot b_{ij}/b_{ik}\}$. Consequently, $x_j = \sum_i x_{ij} \le x_j^\circ + 2 \cdot \sum_i \max_k \frac{b_{ij}}{b_{ik}}$. □

Before proving Lemma 5.2, we state the following claims. We let Δp_j denote $p^{t+1} - p^t$. In the following claims, the t index on the prices and demands is implicit.

CLAIM 5.4. *For all j,*

$$\frac{1}{e_i}\sum_{j,k} x_{ij}x_{ik}\Delta p_j \Delta p_k \le \sum_l \frac{x_{i\ell}}{p_\ell}(\Delta p_\ell)^2.$$

CLAIM 5.5. *Suppose that for all j,* $|\Delta p_j| \le p_j/4$*. Then*

$$\frac{(\Delta p_j)^2}{p_j} \le 6 d_{KL}(p_j + \Delta p_j, p_j).$$

PROOF. (of Lemma 5.2.) We write $\phi(p^t)$ and $\phi(p^{t+1})$ as functions of the p_j, and then upper bound these terms using the inequalities $x(1+x)^{-1} \le x + 2x^2$ for $|x| \le \frac{1}{2}$ and $\log(1+y) \le y$ for $|y| \le 1$, along with Claims 5.4 and 5.5.

$$\begin{aligned}&\phi(p^{t+1}) - \ell_\phi(p^{t+1}) \\ &= \sum_j (p_j + \Delta p_j) - \sum_i e_i \log \sum_k b_{ik}(p_k + \Delta p_k) \\ &\qquad - \sum_j p_j + \sum_i e_i \log \sum_k b_{ik}p_k + \sum_j z_j \Delta p_j \\ &= \sum_j x_j \Delta p_j + \sum_i e_i \log \frac{\sum_k b_{ik}p_k}{\sum_k b_{ik}(p_k + \Delta p_k)} \\ &= \sum_j x_j \Delta p_j + \sum_i e_i \log\left[1 - \frac{\sum_k b_{ik}\Delta p_k}{\sum_k b_{ik}p_k}\left(1 + \frac{\sum_l b_{i\ell}\Delta p_\ell}{\sum_l b_{i\ell}p_\ell}\right)^{-1}\right].\end{aligned}$$

Next we use the bound $x(1+x)^{-1} \le x + 2x^2$ for $|x| \le \frac{1}{2}$, noting that $|\frac{\sum_l b_{i\ell}\Delta p_\ell}{\sum_l b_{ik}p_\ell}| \le \frac{1}{2}$, as every $|\Delta p_\ell| \le \frac{1}{2}p_\ell$ by assumption. Thus:

$$\begin{aligned}\phi(p^{t+1}) - \ell_\phi(p^{t+1}) &\le \sum_j x_j \Delta p_j \\ &+ \sum_i e_i \log\left[1 - \frac{\sum_k b_{ik}\Delta p_k}{\sum_k b_{ik}p_k} + 2\frac{\sum_k b_{ik}\Delta p_k \sum_l b_{i\ell}\Delta p_\ell}{\sum_k b_{ik}p_k \sum_l b_{i\ell}p_\ell}\right].\end{aligned}$$

Now we use the bound $\log(1+y) \le y$, which applies as the second and third terms in the log are each bounded by $\frac{1}{2}$

[9] A more careful argument shows the multiplier of 2 is not needed.

(recall that $|\frac{\sum_l b_{i\ell}\Delta p_\ell}{\sum_l b_{ik}p_\ell}| \le \frac{1}{2}$). Hence:

$$\begin{aligned}
&\phi(p^{t+1}) - \ell_\phi(p^{t+1}) \\
&\le \sum_j x_j \Delta p_j - \sum_i e_i \frac{\sum_k b_{ik}\Delta p_k}{\sum_k b_{ik}p_k} + 2e_i \frac{\sum_k b_{ik}\Delta p_k \sum_l b_{i\ell}p_\ell}{\sum_k b_{ik}p_k \sum_l b_{ik}p_\ell} \\
&\le \sum_j x_j \Delta p_j - \sum_k x_k \Delta p_k + 2\sum_i \frac{1}{e_i}\sum_k x_{ik}\Delta p_k \sum_l x_{i\ell}\Delta p_\ell \\
&\le 2\sum_{i,j} \frac{x_{ij}}{p_j}(\Delta p_j)^2 = 2\sum_j \frac{x_j}{p_j}(\Delta p_j)^2 \quad \text{(by Claim 5.4)} \\
&\le 12 \sum_j x_j d_{\text{KL}}(p_j + \Delta p_j, p_j) \quad \text{(by Claim 5.5)} \quad \Box
\end{aligned}$$

PROOF. (of Claim 5.4.) This result follows by rewriting e_i as $\sum_k x_{ik}p_k$.

$$\begin{aligned}
&e_i \sum_l \frac{x_{il}}{p_\ell}(\Delta p_\ell)^2 \\
&\quad = \sum_l \frac{x_{il}\left(\sum_k x_{ik}p_k\right)}{p_\ell}(\Delta p_\ell)^2 = \sum_{l,k} x_{il}x_{ik}\frac{p_k}{p_\ell}(\Delta p_\ell)^2 \\
&\quad = \sum_l x_{il}^2(\Delta p_\ell)^2 + \sum_{k,l:k\neq l} x_{ik}x_{il}\frac{p_k}{p_\ell}(\Delta p_\ell)^2 \\
&\quad = \sum_l x_{il}^2(\Delta p_\ell)^2 + \sum_{k<l} x_{ik}x_{il}\left(\frac{p_k}{p_\ell}(\Delta p_\ell)^2 + \frac{p_\ell}{p_k}(\Delta p_k)^2\right).
\end{aligned}$$

Now, we apply the AM-GM inequality:

$$\begin{aligned}
e_i \sum_l \frac{x_{il}}{p_\ell}(\Delta p_\ell)^2 &\ge \sum_l x_{il}^2(\Delta p_\ell)^2 + \sum_{k<l} x_{ik}x_{il}\cdot 2|\Delta p_\ell||\Delta p_k| \\
&= \sum_{j,k} x_{ij}x_{ik}|\Delta p_j||\Delta p_k|. \quad \Box
\end{aligned}$$

PROOF. (of Claim 5.5.) We use the bound $\log x \ge x - \frac{2}{3}x^2$ for $|x| \le \frac{1}{4}$.

$$\begin{aligned}
&d_h(p_j + \Delta p_j, p_j) \\
&= (p_j + \Delta p_j)\log(p_j + \Delta p_j) \\
&\quad - (p_j + \Delta p_j) - p_j \log p_j + p_j - (\log p_j)\Delta p_j \\
&= -\Delta p_j + (p_j + \Delta p_j)\log\left(1 + \frac{\Delta p_j}{p_j}\right) \\
&\ge -\Delta p_j + (p_j + \Delta p_j)\left(\frac{\Delta p_j}{p_j} - \frac{2}{3}\frac{(\Delta p_j)^2}{p_j^2}\right) \\
&= \frac{1}{3}\frac{(\Delta p_j)^2}{p_j}\left(1 - 2\frac{\Delta p_j}{p_j}\right) \\
&\ge \frac{1}{6}\frac{(\Delta p_j)^2}{p_j}. \quad \Box
\end{aligned}$$

6. CES UTILITIES

In this section we consider the weighted update rule,

$$p_j^{t+1} = p_j^t e^{(z_j/\gamma_j^t)}, \tag{11}$$

for markets in which every buyer has a complementary CES utility, i.e. the ith buyer has a parameter ρ_i in the range $-\infty < \rho_i < 0$. (The markets in which all buyers have substitutes CES utility functions have been analyzed in previous work.) In addition, the weights γ_j^t are allowed to change from one time step to the next; our updates to price p_j will use the weight $\gamma_j^t = 5\cdot\max\{1, x_j^t\}$.[10] This seems a very natural distributed rule, and indeed a linearization of this rule, $p_j^{t+1} = p_j^t[1 + \lambda\max\{1, z_j\}]$[11] was used in the prior works by Cole et al. [7] and Cheung et al. [4].

For these markets we will show that $\phi(p^t) - \phi(p^*)$ reduces by at least a $1-\mu$ factor at each time step, where $0 < \mu < 1$ depends on the initial price and the market parameters we will specify.

Henceforth, the t index on all the parameters except prices will be implicit.

Notation.

Recall that e_i denotes buyer i's budget, and define

$$c_i := \rho_i/(\rho_i - 1).$$

Note that $c_i = 1 - \sigma_i$, where σ_i is the demand elasticity of the associated CES utility function, and let $c' = \max_i c_i$. Finally, set $\gamma = \max_j \gamma_j$. Again, we let Δp_j denote $p_j^{t+1} - p_j^t$.

As is well known, the demand for good j when buyer i optimizes her utility is given by

$$x_{ij} = e_i b_{ij} p_j^{c_i - 1} S_i^{-1},$$

where $b_{ij} := a_{ij}^{c_i - 1}$ and $S_i = \sum_\ell b_{i\ell}p_\ell^{c_i}$. The optimal utility equals $e_i S_i^{-1/c_i}$.

Now, using Equation (4), we see that ϕ is given by

$$\phi(p^t) = \sum_j p_j^t - \sum_i e_i \log S_i^{1/c_i}.$$

In the next two subsections we show that the the potential function in this case satisfies a stronger sandwiching property, as specified in Lemmas 6.2 and 6.3. This stronger property immediately yields the claimed bound on the convergence rate (Theorem 6.4).

CLAIM 6.1. $|p_j^{t+1} - p_j^t| \le \frac{1}{4}p_j^t$.

PROOF. $|p_j^{t+1} - p_j^t| \le (e^{1/5} - 1)p_j^t \le \frac{1}{4}p_j^t$. $\Box$

LEMMA 6.2. *Suppose that* $|p_j^{t+1} - p_j^t| \le \frac{1}{4}p_j^t$ *for all* j. *Then*

$$\phi(p^t) - \phi(p^{t+1}) \ge \frac{1}{2}\sum_j \frac{z_j^2 p_j^t}{\gamma_j}.$$

LEMMA 6.3.

$$\phi(p^t) - \phi(p^*) \le \max_j\left\{10, \frac{5}{2m_j}\right\}\sum_j \frac{z_j^2 p_j^t}{\gamma_j},$$

where $m_j = \frac{1 - r_j^{c'} + c'(r_j - 1)}{c'(r_j - 1)^2}$ *and* $r_j = p_j^*/p_j^t$.

We can now deduce our main result.

THEOREM 6.4. *For all CES markets, for the sequence of prices* p^t *defined by the update rule* (11)*, for all* t,

$$\phi(p^t) - \phi(p^*) \le [(1 - \Theta(1)]^t d_{KL}(p^*, p^0)$$

In other words, for any $\epsilon > 0$, $\phi(p^t) - \phi(p^*) \le \epsilon d_{KL}(p^*, p^0)$, *if* $t = \Omega(\log(1/\epsilon))$.

[10] Any greater value for γ_j would work too.

[11] The λ replaces the constant of 5 used here, as a greater range of values for this parameter is needed in markets of substitutes.

PROOF.

$$\phi(p^{t+1}) - \phi(p^*) \;=\; \phi(p^t) - \phi(p^*) - [\phi(p^t) - \phi(p^{t+1})]$$

$$\le \phi(p^t) - \phi(p^*) - \frac{1}{2}\sum_j \frac{z_j^2 p_j^t}{\gamma_j} \quad \text{(by Lemma 6.2)}$$

$$\le [\phi(p^t) - \phi(p^*)]\left[1 - \frac{1}{2}\left(\max_j\left\{10, \frac{5}{2m_j}\right\}\right)^{-1}\right] \quad \text{(by Lemma 6.3)}$$

$$= (1 - \Theta(1))[\phi(p^t) - \phi(p^*)]. \quad \square$$

As we will see, m_j is a decreasing function of $r_j = p_j^*/p_j$. Consequently, we will need to show that p_j^*/p_j remains bounded throughout the tatonnement process in order to prove convergence. This is done in Section 6.3.

6.1 The Upper Bound: Good Progress on a Price Update

The proof of Lemma 6.2 proceeds in two steps. First, we show that $\phi(p^{t+1}) - \phi(p^t) + \sum_j z_j[p_j^{t+1} - p_j^t] \le 2\sum_j \frac{x_j}{p_j}[p_j^{t+1} - p_j^t]^2$. We then choose $\gamma_j = 5 \cdot \max\{1, x_j\}$. Finally, we deduce the bound in Lemma 6.2. Our first bound uses the following result.

LEMMA 6.5. *Suppose that for all j, $|\Delta p_j|/p_j \le \frac{1}{4}$. Then $\phi(p + \Delta p) - \ell_\phi(p + \Delta p; p) \doteq \phi(p + \Delta p) - \phi(p)$ $+ \sum_j z_j \Delta p_j \le 2\sum_j \frac{x_j}{p_j}(\Delta p_j)^2$.*

PROOF. As in the proof of Lemma 5.2, we use two bounds: First, a bound on $\log(1 + \epsilon)$, namely:

$$\log(1 + \epsilon) \ge \epsilon - \frac{2}{3}\epsilon^2, \text{ when } |\epsilon| \le \frac{7}{18}. \qquad (12)$$

And second, a bound on the following polynomial, which follows from simple calculus: if $|\Delta p_j/p_j| \le 1/4$ and $0 \le c \le 1$,

$$(p_j + \Delta p_j)^c \ge p_j^c + c p_j^{c-1}(\Delta p_j) - \frac{2}{3} c p_j^{c-2}(\Delta p_j)^2. \qquad (13)$$

We let D_ϕ denote $\phi(p + \Delta p) - \ell_\phi(p + \Delta p; p)$, for short. Recall that $S_i(p) = \sum_\ell b_{i\ell} p_\ell^{c_i}$. Then:

$$D_\phi = \phi(p + \Delta p) - \phi(p) + \sum_j z_j \Delta p_j$$

$$= \sum_j \Delta p_j + \sum_j z_j \Delta p_j - \sum_i \frac{e_i}{c_i} \log \frac{S_i(p + \Delta p)}{S_i(p)}.$$

$$= \sum_j x_j \Delta p_j - \sum_i \frac{e_i}{c_i} \log\left(\frac{\sum_\ell b_{i\ell}(p_\ell + \Delta p_\ell)^{c_i}}{S_i(p)}\right).$$

As $\rho < 0$, $0 < c_i < 1$. So we can apply (13), yielding:

$$D_\phi \le \sum_j x_j \Delta p_j - \sum_i \frac{e_i}{c_i} \log\left(1 + \frac{\sum_\ell b_{i\ell} c_i p_\ell^{c_i - 1}(\Delta p_\ell)}{S_i(p)} - \frac{\frac{2}{3}\sum_\ell b_{i\ell} c_i p_\ell^{c_i - 2}(\Delta p_\ell)^2}{S_i(p)}\right).$$

Recalling that $x_{i\ell} = e_i b_{i\ell} p_\ell^{c_i - 1}/S_i(p)$, yields:

$$D_\phi \le \sum_j x_j \Delta p_j - \sum_i \frac{e_i}{c_i} \log\left(1 + \sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta p_\ell) - \frac{2}{3}\sum_\ell c_i \frac{x_{i\ell}}{p_\ell e_i}(\Delta p_\ell)^2\right).$$

On applying (12), and noting that $\sum_\ell x_{i\ell} p_\ell \le e_i$, $c_i \le 1$, and $|\Delta p_\ell|/p_\ell \le \frac{1}{4}$, we obtain the bound:

$$D_\phi$$

$$\le \sum_j x_j \Delta p_j - \sum_i \frac{e_i}{c_i}\left(\sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta p_\ell) - \frac{2}{3}\sum_\ell \frac{c_i x_{i\ell}}{p_\ell e_i}(\Delta p_\ell)^2\right)$$

$$+ \sum_i \frac{e_i}{c_i}\frac{2}{3}\left(\sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta p_\ell) - \frac{2}{3}\sum_\ell c_i \frac{x_{i\ell}}{p_\ell e_i}(\Delta p_\ell)^2\right)^2$$

$$= \frac{2}{3}\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2$$

$$+ \frac{2}{3}\sum_i \frac{c_i}{e_i}\left(\sum_\ell x_{i\ell}(\Delta p_\ell) - \frac{2}{3}\sum_\ell \frac{x_{i\ell}}{p_\ell}(\Delta p_\ell)^2\right)^2$$

$$= \frac{2}{3}\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{2}{3}\sum_i \frac{c_i}{e_i}\left(\sum_\ell x_{i\ell}(\Delta p_\ell)\left(1 - \frac{2\Delta p_\ell}{3p_\ell}\right)\right)^2.$$

Now recall that $\Delta p_\ell/p_\ell \le \frac{1}{4}$, to give the bound:

$$D_\phi \le \frac{2}{3}\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{2}{3}\sum_i \frac{c_i}{e_i}\left(\sum_\ell x_{i\ell}|\Delta p_\ell| \cdot \frac{7}{6}\right)^2$$

$$= \frac{2}{3}\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{49}{54}\sum_i \frac{1}{e_i}\left(\sum_\ell x_{i\ell}|\Delta p_\ell|\right)^2$$

$$= \frac{2}{3}\sum_\ell \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 + \frac{49}{54}\sum_i \frac{1}{e_i}\sum_{j,k} x_{ij} x_{ik}|\Delta p_j||\Delta p_k|$$

$$\le \left(\frac{2}{3} + \frac{49}{54}\right)\sum \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2 \quad \text{(by Claim 5.4)}$$

$$\le 2\sum \frac{x_\ell}{p_\ell}(\Delta p_\ell)^2. \quad \square$$

PROOF. (of Lemma 6.2.) Recall that $\Delta p_j = p_j^{t+1} - p_j^t$ and that $p_j^{t+1} = p_j^t e^{(z_j/\gamma_j)}$. By Lemma 6.5,

$$\phi(p^t) - \phi(p^{t+1}) \;\ge\; \sum_j z_j[p_j^{t+1} - p_j^t] - 2\sum_j \frac{x_j}{p_j^t}[p_j^{t+1} - p_j^t]^2.$$

Next, using the formula for p^{t+1} and the fact that $\gamma_j \ge 5x_j$ gives the bound:

$$\phi(p^t) - \phi(p^{t+1})$$

$$\ge \sum_j z_j p_j^t[e^{(z_j/\gamma_j)} - 1] - \frac{2}{5}\sum_j \gamma_j p_j^t[e^{(z_j/\gamma_j)} - 1]^2$$

$$\ge \sum_j z_j p_j^t[e^{(z_j/\gamma_j)} - 1]\left(1 - \frac{2}{5}\frac{\gamma_j}{z_j}[e^{(z_j/\gamma_j)} - 1]\right)$$

$$\ge \sum_{z_j \ge 0} \frac{z_j^2}{\gamma_j}\left(1 - \frac{2}{5}\cdot\frac{10}{9}\right) + \sum_{z_j < 0} \frac{z_j^2}{\gamma_j}\frac{9}{10}\left(1 - \frac{2}{5}\right)$$

$$\ge \frac{1}{2}\sum_j \frac{z_j^2 p_j^t}{\gamma_j}. \quad \square$$

6.2 Upper Bound on Distance to Equilibrium

Before proving Lemma 6.3, we list a few facts which can be easily proved by calculus.

FACT 6.6. *i. For $0 < c < 1$, $h_c(r) := \frac{1 - r^c + c(r-1)}{(r-1)^2}$ is a decreasing function of r.*

ii. $h_c(r)/c$ *is a decreasing function of* c.

iii. If $|s| \le 1$, *then* $e^s - 1 \ge s + s^2/3$.

LEMMA 6.7. *Suppose that* $p_j^*/p_j \le r_j$ *for all* j, *where* $r_j \ge 1$. *Let* $c' = \max_i c_i$. *Then*

$$\phi(p^*) - \ell_\phi(p^*;p) \ge \sum_\ell \frac{h_{c'}(r_\ell)}{c'} x_\ell \cdot \frac{(p_\ell^* - p_\ell)^2}{p_\ell}.$$

PROOF. As with previous lemmas, we use a bound on the polynomial $(p_j^* - p_j)^{c_i}$, but now we use the bound given by Fact 6.6i. Specifically, if $p_j^*/p_j \le r_j$ and $0 < c \le 1$, $(p_j^*)^c \le p_j^c + cp_j^{c-1}(p_j^* - p_j) - h_c(r_j)p_j^{c-2}(p_j^* - p_j)^2$. We also use a simple bound on the log function, namely $\log(1+\epsilon) \le \epsilon$ for $\epsilon \ge -1$. To avoid clutter, we omit the superscript t on the prices.

Let $\Delta^* p_j = p_j^* - p_j$. Then

$$\phi(p^*) - \ell_\phi(p^*;p) = \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i} \log\left(\frac{\sum_\ell b_{i\ell}(p_l^*)^{c_i}}{S_i(p)}\right).$$

Using the upper bound on $(p_j^*)^{c_i}$ gives:

$$\begin{aligned} &\phi(p^*) - \ell_\phi(p^*;p) \\ &\ge \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i} \log\left(1 + \frac{\sum_\ell b_{i\ell} c_i p_\ell^{c_i - 1}(\Delta^* p_\ell)}{S_i(p)} \right. \\ &\qquad\qquad \left. - \frac{\sum_\ell b_{i\ell} h_{c_i}(r_\ell) p_\ell^{c_i-2}(\Delta^* p_\ell)^2}{S_i(p)}\right) \\ &= \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i} \log\left(1 + \sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta^* p_\ell) \right. \\ &\qquad\qquad \left. - \sum_\ell h_{c_i}(r_\ell) \frac{x_{i\ell}}{p_\ell e_i}(\Delta^* p_\ell)^2\right) \end{aligned}$$

On noting that the argument for the log is positive (as it is an upper bound for p_j^*), we can apply the bound $\epsilon \ge \log(1+\epsilon)$ for $\epsilon \ge -1$ to give:

$$\begin{aligned} &\phi(p^*) - \ell_\phi(p^*;p) \\ &\ge \sum_j x_j \Delta^* p_j - \sum_i \frac{e_i}{c_i} \left(\sum_\ell c_i \frac{x_{i\ell}}{e_i}(\Delta^* p_\ell) \right. \\ &\qquad\qquad \left. - \sum_\ell h_{c_i}(r_\ell) \frac{x_{i\ell}}{p_\ell e_i}(\Delta^* p_\ell)^2\right) \\ &= \sum_i \sum_\ell \frac{h_{c_i}(r_\ell)}{c_i} x_{i\ell} \frac{(\Delta^* p_\ell)^2}{p_\ell} \\ &\ge \sum_i \sum_\ell \frac{h_{c'}(r_\ell)}{c'} x_{i\ell} \frac{(\Delta^* p_\ell)^2}{p_\ell} \qquad \text{(by Fact 6.6 ii.)} \\ &= \sum_\ell \frac{h_{c'}(r_\ell)}{c'} x_\ell \frac{(\Delta^* p_\ell)^2}{p_\ell}. \quad \square \end{aligned}$$

PROOF. (of Lemma 6.3.) Let m_j denote $h_{c'}(r_j)/c'$. Then, by Lemma 6.7:

$$\begin{aligned} \phi(p^t) - \phi(p^*) &\le \sum_j z_j(p_j^* - p_j^t) - \sum_j m_j x_j \frac{(p_j^* - p_j^t)^2}{p_j^t)} \\ &\le \max_{p'} \sum_j \left(z_j(p_j' - p_j^t) - m_j x_j \frac{(p_j' - p_j^t)^2}{p_j^t}\right). \end{aligned}$$

There are two cases.

Case 1: $0 \le x_j \le 1/2$.

Then $-1 \le z_j \le -1/2$ and hence $z_j \ge -2z_j^2$.

$$z_j(p_j' - p_j^t) - m_j x_j \frac{(p_j' - p_j^t)^2}{p_j^t} \le -z_j p_j^t \le 2z_j^2 p_j^t = 2\gamma_j \frac{z_j^2 p_j^t}{\gamma_j}.$$

As $x_j \le 1/2 < 1$, $2\gamma_j = 10$. Hence

$$z_j(p_j' - p_j^t) - m_j x_j \frac{(p_j' - p_j^t)^2}{p_j^t} \le 10 \frac{z_j^2 p_j^t}{\gamma_j}.$$

Case 2: $x_j \ge 1/2$.

$z_j(p_j' - p_j^t) - m_j x_j \frac{(p_j' - p_j^t)^2}{p_j^t}$ is a quadratic function of $(p_j' - p_j^t)$. The quadratic function is maximized when $(p_j' - p_j^t) = \frac{z_j p_j^t}{2m_j x_j}$, with its maximum value being $\frac{z_j^2 p_j^t}{4m_j x_j} = \frac{\gamma_j}{4m_j x_j} \frac{z_j^2 p_j^t}{\gamma_j}$.

As $x_j \ge 1/2$ and $\gamma_j = 5 \cdot \max\{1, x_j\}$, $\gamma_j / x_j \le 10$. Hence

$$z_j(p_j' - p_j^t) - m_j x_j \frac{(p_j' - p_j^t)^2}{p_j^t} \le \frac{5}{2m_j} \frac{z_j^2 p_j^t}{\gamma_j}.$$

Combining the two cases yields the result. $\square$

6.3 Bounding m_j

Let $p_U = \max_j\{p_j^\circ\}$, the maximum initial price, $U = \max\{p_U, M\}$, and $L^* = \min_j\{p_j^*\}$.

LEMMA 6.8. $p_j^*/p_j^t \le \max\{2p_j^*/p_j^\circ, 2(L^*/2U)^{\min_i \rho_i}\}$

PROOF. *Observation 1.* No price will exceed $2U$ during the entire tatonnement process.

Reason. Suppose not, then let $t = \tau$ be the first time when some price, say p_k, exceed $2U$. At $t = \tau - 1$, $p_k^{\tau-1} < 2U$. But $p_k^{\tau-1} \ge U$, as p_k can at most double in one time unit.

Then $p_k^{\tau-1} \ge M$ and $x_k^{\tau-1} \le M/p_k^{\tau-1} \le 1$. By the price update rule, $p_k(\tau) \le p_k^{\tau-1} < 2U$, a contradiction.

Observation 2. $p_k \ge \frac{1}{2} \cdot \min\{p_k^\circ, (2U/L^*)^{\min_i \rho_i} p_k^*\}$ throughout the entire tatonnement process.

Reason. Suppose that for some k, $p_k \le L^*(2U/L^*)^{\min_i \rho_i}$. We claim that $x_k \ge 1$. At equilibrium prices, all demands equal 1. If the prices are all raised by a factor of $\frac{2U}{L^*}$, then all demands equal $\frac{L^*}{2U}$. Note that now all prices are at least $2U$.

Now reduce the price of p_k from $\frac{2U}{L^*} p_k^*$ to $\left(\frac{2U}{L^*}\right)^{\min_i \rho_i} p_k^*$, that is, reduce the price by a factor of $\left(\frac{2U}{L^*}\right)^{1-\min_i \rho_i}$. By the elasticity bound, the new demand for good k is

$$x_k' \ge x_k \cdot \left[\left(\frac{2U}{L^*}\right)^{1-\min_i \rho_i}\right]^{1/(1-\min_i \rho_i)} = \frac{L^*}{2U} \cdot \frac{2U}{L^*} = 1.$$

We just proved that when $p_k = \left(\frac{2U}{L^*}\right)^{\min_i \rho_i} p_k^*$ but all other prices are at least $2U$, the demand for good k is at least 1. By Observation 1, no price exceeds $2U$ during the entire tatonnement process. In complementary markets, since the demand for one good increases when the prices of other goods decrease, we have shown that $x_k \ge 1$ throughout the entire tatonnement process, if $p_k \le \left(\frac{2U}{L^*}\right)^{\min_i \rho_i} p_k^*$.

Let $\bar{L}_k = (1/2) \cdot \min\{p_k^\circ, (2U/L^*)^{\min_i \rho_i} p_k^*\}$. Suppose that Observation 2 were incorrect, then let $t = \tau$ be the first time when some price, say p_j, is below $\bar{L}_j$. At $t = \tau - 1$,

$p_j^{\tau-1} \geq \bar{L}_j$. But $p_j^{\tau-1} \leq 2\bar{L}_j$, as p_j can reduce by at most half in one time unit.

Then $x_j^{\tau-1} \geq 1$. By the price update rule, $p_j^{\tau} \geq p_j^{\tau-1} \geq \bar{L}_j$, a contradiction.

The lemma now follows from Observation 2. □

7. DISCUSSION

We have shown that discrete versions of tatonnement converge for Leontief and CES utilities. The main open question is whether these convergence results extend to the Ongoing Market model defined by Cole and Fleisher [7]. In this model, the market repeats from one time period to the next, and excess demands and supplies are carried foward to successive time periods using finite buffers, which they called warehouses. The purpose of this model was to provide a more natural setting for the tatonnement update process.

There are two aspects to the Ongoing Market that our results do not address.

- Warehouses. There is a separate warehouse for each good. The price update for each good is adjusted to take account of whether the warehouse is relatively full or empty. The goal is to show that, as in [4], the tatonnement price update converges to the equilibrium prices and that in addition this can be achieved without having the warehouse either overflow or run out of stock, and further that it too converges to an ideal state, namely half-full. We conjecture that this is possible for the markets with CES utilities at least.

- Asynchrony. This allows the prices to be updated independently, at separate times, with the sole constraint that each price update at least once per time unit. Further, each price update uses the accumulated demand since the previous update, as opposed to the instantaneous demand, to determine its size. Again, both the asynchrony itself, and the price update rule, are intended to provide a process that seems more natural. We also conjecture that this variant of the price update will converge for markets with CES utilities.

The previous analyses for the Ongoing Market used non-trivial amortized arguments. It seems they will not extend to the present setting, for they were intrinsically linear, whereas the potential function employed here for the CES utilities is quadratic. Still, we suspect there may be extensions of the current analyses that will lead to the conjectured results.

Finally, we ask whether the logarithmic rate of convergence extends to the markets with Leontief utilities.

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9. REFERENCES

[1] K. Arrow, H. Block, and L. Hurwicz. On the stability of the competitive equilibrium: II. *Econometrica*, 27(1):82–109, 1959.

[2] Benjamin Birnbaum, Nikhil Devanur, and Lin Xiao. Distributed algorithms via gradient descent for fisher markets. In *ACM Conference on Electronic Commerce*, pages 127–136, 2011.

[3] G. Chen and M. Teboulle. Convergence analysis of a proximal-like minimization algorithm using Bregman functions. *SIAM Journal on Optimization*, 3(3):538–543, August 1993.

[4] Yun Kuen Cheung, Richard Cole, and Ashish Rastogi. Tatonnement in ongoing markets of complementary goods. In *ACM Conference on Electronic Commerce*, pages 337–354, 2012.

[5] Bruno Codenotti, Benton Mccune, and Kasturi Varadarajan. Market equilibrium via the excess demand function. In *Proceedings of STOC*, pages 74–83, 2005.

[6] Bruno Codenotti and Kasturi Varadarajan. Efficient computation of equilibrium prices for markets with leontief utilities. In *Proceedings of the Thirty First International Colloquium on Automata, Languages and Programming*, 2004.

[7] Richard Cole and Lisa Fleischer. Fast-converging tatonnement algorithms for one-time and ongoing market problems. In *STOC*, pages 315–324, 2008.

[8] Nikhil R Devanur. Fisher markets and convex programs. Working paper available from http://research.microsoft.com/en-us/um/people/nikdev/, 2009.

[9] Edmund Eisenberg and David Gale. Consensus of subjective probabilities: The pari-mutuel method. *Ann. Math. Statist.*, 30(1):165–168, 1959.

[10] Lisa Fleischer, Rahul Garg, Sanjiv Kapoor, Rohit Khandekar, and Amin Saberi. A fast and simple algorithm for computing market equilibria. In *WINE*, pages 19–30, 2008.

[11] George Green. *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.* Nottingham, England: T. Wheelhouse, 1828.

[12] Kamal Jain and Vijay V. Vazirani. Eisenberg-Gale markets: algorithms and game-theoretic properties. *Games and Economic Behavior*, 70(1), 2010.

[13] Kamal Jain, Vijay V. Vazirani, and Yinyu Ye. Market equilibria for homothetic, quasi-concave utilities and economies of scale in production. In *SODA*, pages 63–71, 2005.

[14] F. P. Kelly and V. V. Vazirani. Rate control as a market equilibrium. Manuscript, 2002.

[15] Andreu Mas-Collel, Michael D. Whinston, and Jerry R. Green. *Microeconomic Theory.* Oxford University Press, 1995.

[16] H. Nikaido and H. Uzawa. Stability and non-negativity in a Walrasian process. *International Econ. Review*, 1:50–59, 1960.

[17] Bernard Riemann. *Grundlagen fÃijr einen allgemeine Theorie der Functionen einer verÃd'nderlichen complexen GrÃűsse (Basis for a general theory of functions of a variable complex quantity).* GÃűttingen, (Germany): Adalbert Rente, 1867.

[18] H. Scarf. Some examples of global instability of the competitive equilibrium. *International Econ Review*, 1:157–172, 1960.

[19] Léon Walras. *Eléments d'Economie Politique Pure.* Corbaz, Lausanne, 1874. (Translated as: *Elements of Pure Economics.* Homewood, IL: Irwin, 1954.).