On Competitiveness in Uniform Utility Allocation Markets ⋆

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Abstract. In this paper, we study competitive markets - a market is competitive if increasing the endowment of any one buyer does not increase the equilibrium utility of any other buyer. In the Fisher setting, competitive markets contain all markets with weak gross substitutability (WGS), a property which enable efficient algorithms for equilibrium computation.

We show that every uniform utility allocation (UUA) market which is competitive, is a submodular utility allocation (SUA) market. Our result provides evidence for the existence of efficient algoritheorems for the class of competitive markets.

1 Introduction

In the past few years, there has been a surge of activity to design efficient algorithms for computation of market equilibrium. These include the linear utilities case in the Fisher model [9, 11] and the Arrow-Debreu model [13], the spending constraint model [10], Leontief utility functions in the Fisher model [8] and so on. Interestingly, almost all of these markets for which efficient equilibrium computation algorithms are known, satisfy the property of *weak gross substitutability* (WGS). A market is WGS if raising the price of any good does not lead to the decrease in the demand of some other good. This property has extensively been studied in mathematical economics, [1, 16, 2] and recently Codenotti et.al. [7] gave polytime algorithms to compute equilibriums in WGS markets, under fairly general assumptions.

WGS relates how one good's price influences the demands for other goods. Analogously, competitiveness relates how one person's assets influence the returns to others. A market is called competitive if increasing the money of one agent cannot lead to increase in the equilibrium utility of some other agent. This notion was introduced by Jain and Vazirani [14] [1], who showed that in the Fisher setting any WGS market is competitive. In this paper, we provide a characterization of competitive markets in a class of Eisenberg-Gale markets, introduced by [14]. Combined with results of [14], our result provides some evidence that

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[1] They used the term *competition monotonicity* instead

competitive markets, like WGS markets, might also be amenable to efficient algorithms. In particular, [12, 14] gave combinatorial polynomial time algorithms for some markets that were not WGS; our result shows that these markets are competitive.

Recently, Jain and Vazirani [14] proposed a new class of markets called *Eisenberg-Gale markets* or simply EG markets. In 1959, Eisenberg and Gale [11] gave a convex program for obtaining the equilibrium in the linear utilities case of the Fisher model. An EG market is any market whose equilibrium is captured by a similar convex program. Thus, the linear utilities case of Fisher is an example of an EG market. [14] showed that this class captured many other interesting markets including several variants of resource allocation markets defined by Kelly [15] to model TCP congestion control.

The convex program capturing equilibria of EG markets maximizes the money weighted geometric mean of the utilities of buyers over all *feasible* utilities, which form a convex set. For instance, in the program of Eisenberg and Gale [11], the set of feasible utilities are those implied by the condition that no good is oversold. Thus EG markets do away with the concept of goods and deal only with allocations of utility and one can think of EG markets as *utility allocation markets.*

If the constraints on feasible utilities are just those which limit the total utility obtainable by any set of agents, the EG market so obtained is called an *uniform utility allocation (UUA) market.* The linear utilities case of Fisher with the utility of each unit of good for each agent being either 0 or 1 is a UUA market. UUA markets can be represented via a set-function called the *valuation function*, where the value of any subset of agents denotes the maximum utility obtainable by that set. If the valuation function is submodular, the market is called a *submodular utility allocation (SUA) market.* In fact, the Fisher example above turns out to be a SUA market.

[14] define the notion of *competition monotonicity* which we call *competitiveness* in this paper. In their paper, [14] prove that every SUA market is competitive. They also give an algorithm for computing equilibrium in SUA markets. The paper also asks if there exist competitive UUA markets which are not SUA.

Our results:

Our main result, proved in Section 3, answers the question asked in [14]. We show that *any* UUA market which is competitive must be an SUA market. Our characterization of competitive markets in UUA markets shows that the algorithm of [14] works for *all* competitive UUA markets. A natural question is whether there are efficient algorithms for all competitive markets. [14] showed that all WGS markets are competitive and [7] gave efficient algorithms for all WGS markets; this probably gives evidence in favor of existence of algorithms for competitive markets. Competitiveness seems to be a natural property for markets, but a lot remains to understand it clearly. A first step might be investigating competitiveness in EG markets alone. We do not know of any EG markets which are competitive and have irrational equilibria. Apart from SUA markets, the other large class of competitive markets are EG[2] markets [6]: EG markets with only two agents. [6] showed recently that these markets also have rational equilibria. Are all competitive EG markets rational? Settling these questions seems to be an important avenue for research.

Our techniques: We prove all competitive UUA markets are SUA by proving the contrapositive: For *every* UUA market which is not an SUA market, we construct money vectors such that on increasing the money of one particular buyer, the equilibrium utility of some other buyer increases. The main difficulty in constructing these money vectors is that the equilibrium utilities are obtained via solving a convex program with the money as parameters. A change in the money of even one buyer, in general, can change the utilities of all agents.

To argue about the equilibrium utilities, as we see in Section 2, we deal with dual variables, the prices for various subsets of agents, which act as a certificates to equilibrium utility allocations. We use the *non-submodularity* of the valuation function to identify the precise set of agents having money, and the precise amount of money to be given to them. As we see, this construction is delicate, and in particular requires proving the following fact about non-submodular functions which might be of independent interest. This is the most technical part of the paper and is proved in Section 4

Given an allocation, call a set of agents *tight* (w.r.t the allocation) if the total utility of agents in that set equals the maximum allowed by the valuation function. If a valuation function v is not submodular, then there exists a set of agents T,agents $i, j \notin T$, and a feasible utility allocation so that

1. $T, T \cup i, T \cup j$ are tight.
2. *No set* containing both i and j are tight.
3. *All* tight sets containing i or j also contain a common agent l.

The correctness of the algorithm of [14] for finding equilibria in SUA markets and thus our proof that SUA markets are competitive, use crucially the fact that if v is submodular, tight sets formed are closed under taking unions or intersections. Note that this implies if v is submodular, conditions 1 and 2 cannot hold simultaneously

Relation to combinatorial auctions: A result similar to ours is in the setting of combinatorial auctions: Ausubel and Milgrom [3, 4] show that if goods are WGS, then the *coalitional value function* (the maximum value a set of buyers can obtain by forming a coalition) is submodular. A coalitional value function can be used to define a UUA market in a natural way. Further, all WGS markets are competitive. So both the concepts of UUA markets and competitiveness, and in turn our results are more general than the aforementioned.

2 Preliminaries

Definition 1. *An EG market $\mathcal{M}$ with agents $[n]$ is one where the feasible utilities $u \in \mathbf{R}_+^\mathbf{n}$ of the agents can be captured by a polytope*

$$\mathcal{P} = \{\forall j \in J : \quad \sum_{i \in [n]} a_{ij} u(i) \leq b_j \qquad u(i) \geq 0\}$$

with the following free disposal *property: If u is a feasible utility allocation, then so is any u' dominated by u.*

Remark: The definitions in [14, 6] also include auxiliary variables in the definition, but the above is an equivalent definition which will be sufficient for the purposes of this paper.

Example: By definition and the result of Eisenberg and Gale [11], the Fisher market with linear utilities is an EG market. Another example is the following resource allocation market defined by Kelly. Given a network, agents own source-sink pairs and wish to buy capacities on edges so as to send flows from source to sink. The utility $u(i)$ of each agent is the amount of flow it sends. The various flow vectors are constrained via capacity constraints on each edge, which form the convex flow polytope $\mathcal{P}$ above.

An instance of an EG market $\mathcal{M}$ is given by the money of the agents $m \in \mathbf{R}_+^\mathbf{n}$. The equilibrium utility allocation of an EG market is captured by the following convex program similar to the one considered by Eisenberg and Gale [11] for the Fisher market with linear utilities.

$$\max \sum_{i=1}^{n} m_i \log u(i) \quad \text{ s.t. } \quad u \in \mathcal{P}$$

Since the objective function is strictly concave and $\mathcal{P}$ is non-empty, the equilibrium always exists and is unique. Applying the Karash-Kuhn-Tucker (KKT) conditions (see e.g. [5]) characterizing optima of convex programs, for each constraint we have a Lagrangean variable $\mathbf{p_j}$ which we think of as *price* of the constraint, and we have the following equivalent definition of equilibrium allocations in EG markets.

Definition 2. *Given a market instance $m \in \mathbf{R}_+^\mathbf{n}$ of an EG market $\mathcal{M}$, a feasible utility allocation $u \in \mathbf{R}_+^\mathbf{n}$ is an* equilibrium allocation *if there exists* prices $p \in \mathbf{R}_+^{|\mathbf{J}|}$ *satisfying*

- *For all agents $i \in [n]$, $m_i = u(i) \cdot rate(i)$ where $rate(i) = (\sum_{j \in J} a_{ij} p(j))$, the money spent by agent i to get unit utility.*
- $\forall j \in J : p(j) > 0, \qquad \sum_{i \in [n]} a_{ij} u(i) = b_j$

Thus, in the equilibrium allocation, only those constraints are priced which are satisfied with equality (these constraints are called tight constraints), and each agent exhausts his or her money paying for the utility he obtains.

Example In the case of the Fisher setting with linear utilities, there is a constraint for each good and the prices exactly correspond to the unit price of the good. In the resource allocation market described above, there is a price for each edge. Price of an edge is non-zero only if it is saturated by the various flows and each agent exhausts his or her money buying the capacities on edges.

We now consider the case when each a_{ij} above is either 0 or 1.

Definition 3. *An EG market $\mathcal{M}$ is a UUA market if the feasible region $\mathcal{P}$ of utilities can be encoded via a* valuation function $v : 2^{[n]} \to \mathbf{R}$ *as follows*

$$\mathcal{P} = \{\forall S \subseteq [n] \quad \sum_{i \in S} u(i) \leq v(S)\}$$

Such an EG market will be denoted as $\mathcal{M}(v)$, as the market constraints is completely described by v.

Definition 4. *If the valuation function v in Definition 3 is a submodular function, then the market is called a* Submodular Utility Allocation *(SUA) market. To remind, a function $v : 2^{[n]} \to \mathbf{R}$ is submodular if for all sets $S, T \subseteq [n]$, $v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$.*

Example It is not hard to see if in the resource allocation market above, all the agents have the same source, then the coefficients in the flow polytope description are all 0 or 1, implying the market is UUA. In fact, [14] show that this market is SUA as well.

For UUA (and SUA) markets, as in Definition 2 the following gives a characterization of the equilibrium allocation. Given a feasible utility allocation u, a set S is called *tight* if $u(S) \equiv \sum_{i \in S} u(i) = v(S)$.

Definition 5. *For a UUA market, an utility allocation u is the equilibrium allocation iff there exists prices for each subset $S \subseteq [n]$ such that*

- $\forall S \subseteq [n], p(S) > 0 \Rightarrow S$ *is tight.*
- *For all $i \in [n]$, $m_i = u(i) \cdot rate(i)$ where $rate(i) = \sum_{i \in S} p(S)$.*

Given a UUA market, the following observation of [14] shows assumptions we can make on the valuation function. For completeness, we give a proof of the lemma below in the appendix.

Lemma 1. *The valuation function v of UUA markets can be assumed to have the following properties*

- Non degeneracy*: $v(\varnothing) = 0$*
- Monotonicity*: $S \subseteq T \Rightarrow v(S) \leq v(T)$*

– Non redundancy of sets*: For any subset of agents $T \subseteq [n]$, there exists a feasible utility allocation u such that $\sum_{i \in T} u(i) = v(T)$.*
– Complement free*: $v(S \cup T) \leq v(S) + v(T)$.*

Proof. Given the valuation function v define the *covering closure* v^* of v as the follows. For a set $S \subseteq [n]$,

$$v^*(S) = \{\min \sum_{T \subseteq [n]} v(T) x_T \ \text{ s.t } \ \forall i \in S \sum_{T: i \in T} x_T \geq 1 \quad x_T \geq 0\}$$

We prove v^* satisfies all the properties above and v^* is feasible iff v is feasible.

It is easy to see from definition that v^* is non degenerate and monotone. By LP duality we have the following characterization of v^*. For any set $S \subseteq [n]$,

$$v^*(S) = \{\max \sum_{i \in S} u(i) \ \text{ s.t } \ \forall T \subseteq [n] \sum_{i \in T} u(i) \leq v(T) \quad \forall i,\ u(i) \geq 0\}$$

We deliberately use the variable $u(i)$ in the dual. Note that any feasible allocation for v is a feasible solution to the above. This shows that if u is feasible for v, then it is also feasible for v^*.

Non redundancy follows from the fact that for each set S, the dual variables corresponding to the above program gives a feasible allocation with the property $v^*(S) = \sum_{i \in S} u(i)$. Complement free follows from non redundancy as follows: there exists feasible allocation u such that $v^*(S \cup T) = \sum_{i \in S \cup T} u(i) \leq \sum_{i \in S} u(i) + \sum_{i \in T} u(i) \leq v^*(S) + v^*(T)$, where the last inequality follows via feasibility.

Note that the non-redundancy condition above implies for every set S, there exists a feasible allocation which makes it tight. We now define competitiveness.

Definition 6. ([14])
An EG market $\mathcal{M}$ is competitive *(competition monotone in [14]) if for any money vector m, any agent $i \in [n]$ and all $\epsilon > 0$, let u, u' be the equilibrium allocations with money m and m', where $m'(j) = m(j)$ for all $j \neq i$ and $m'(i) = m(i) + \epsilon$, we have $u'(j) \leq u(j)$ for all $j \neq i$.*

In Section 3, we prove the main result of this paper.

Theorem 1. *If a UUA market is competitive, then it is an SUA market.*

3 Competitive UUA markets are SUA markets

In this section we prove Theorem 1. First we state a property we need for non-submodular functions, which we prove in Section 4.

Theorem 2. *Given any valuation function v satisfying the conditions of Lemma 1 which is not submodular, there exists set T, i, j and a feasible utility allocation u such that*

1. $T, T \cup i, T \cup j$ *are tight.*
2. No set *containing both* i *and* j *is tight*
3. All tight sets containing either i *or* j *contain a common element* l *with* $u(l) > 0$.

Proof of Theorem 1 : Let $\mathcal{M}$ be any UUA market which is not an SUA market. We construct money vectors m_1 and m_2 along with the respective equilibrium utility allocations u_1 and u_2, with the following properties:

– $m_2(i) \geq m_1(i)$ for all $i \in [n]$
– There exists j with $m_2(j) = m_1(j)$ and $u_2(j) > u_1(j)$

We first show the above contradicts competitiveness. Since m_2 is greater than m_1 in each coordinate, we can construct vectors $m'_1, m'_2, \cdots, m'_k$ for some k, such that $m'_1 = m_1$, $m'_k = m_2$ and each consecutive m'_i, m'_{i+1} differ in exactly one coordinate j' with $m'_{i+1}(j') > m'_i(j')$. Note that $m'_i(j) = m_1(j) = m_2(j)$.

Let $u'_1, u'_2, \cdots, u'_k$ be the equilibrium allocations corresponding to the money vectors. We have $u_1 = u'_1$ and $u_2 = u'_k$. $u_2(j) > u_1(j)$ implies for some consecutive $i, i+1$ also $u'_{i+1}(j) > u'_i(j)$. Since $m'_{i+1}(j) = m'_i(j)$, we get the contradiction.

To construct the vectors m_1, m_2, we need the structural theorem 2. Let T, i, j, l, u be as in the theorem. To construct both the instances, we first construct feasible utilities and then derive the money vectors such that the allocation are indeed equilibrium utility allocations.

Let $u_1 := u$ except $u_1(i) = 0$. Define $m_1(k) = u_1(k)$ for all k. By condition 1 in Theorem 2, we get $T \cup j$ is tight. Pricing $p(T \cup j) = 1$ shows u_1 is the equilibrium allocation with respect to m_1.

Let $u_2 := u$ except $u_2(i) = u(i) + \epsilon$, $u_2(j) = u(j) + \epsilon$ and $u_2(l) = u(l) - \epsilon$ for some $\epsilon > 0$. ϵ is picked to satisfy two properties: $(a)\epsilon \leq u(l)/2$ and $(b)u_2$ is feasible. We show later how to pick ϵ. Construct m_2 as follows. Define $p' := u_1(j)/u_2(j)$. $m_2(j) = m_1(j)$, $m_2(k) = (2 + p')u_2(k)$ for all $k \in T$, and $m_2(i) = u_2(i)$. Check that m_2 dominates m_1 in each coordinate and $m_2(j) = m_1(j)$.

To see u_2 is an equilibrium allocation w.r.t m_2, note that $T \cup i$, $T \cup j$ remain tight. Let $p(T \cup i) = 2$, $p(T \cup j) = p'$. Check all the conditions of Definition 5 are satisfied.

The proof is complete via the definition of ϵ. Note that in the allocation u_2, the sets which have more utility than in u are ones which contain i or j. By conditions of Theorem 2, one can choose ϵ small enough so that u_2 doesn't make any new set tight and is smaller than $u(l)/2$. To be precise, let

$$\epsilon_i := \min_{Z \subseteq T : Z \cup i \text{ not tight}} (v(Z \cup i) - u(Z \cup i))$$

$$\epsilon_j := \min_{Z \subseteq T : Z \cup j \text{ not tight}} (v(Z \cup j) - u(Z \cup j))$$

$$\epsilon_{ij} := \min_{Z \subseteq T} \frac{v(Z \cup i \cup j) - u(Z \cup i \cup j)}{2}$$

Note by definition $\epsilon_i, \epsilon_j > 0$, and by condition 3 above, $\epsilon_{ij} > 0$. Choose $\epsilon := \min(\epsilon_i, \epsilon_j, \epsilon_{ij}, u(l)/2)$. Again $\epsilon > 0$ which completes the proof of Theorem 1. □

4 A property of non-submodular set functions

In this section we prove the technical theorem 2 about non-submodular functions.

Proof of Theorem 2 : Since v is not submodular, there exists a set $S \cup j$ contradicting submodularity. That is, there is a set S, an agent $j \notin S$ and a strict subset $T \subsetneq S$, such that the marginal value of j for S is greater than that for T. That is, $v(S \cup j) - v(S) > v(T \cup j) - v(T)$

Choose $S \cup j$ to be the smallest set contradicting submodularity. Choose T to be the subset of S for which $v(T \cup j) - v(T)$ is minimum. T, j are that of the theorem.

Since S is the smallest, the restriction of v to every subset of $S \cup j$, in particular T is submodular. Since T is chosen to make marginal of j the minimum, T would have cardinality exactly 1 less than that of S. That is $S = T \cup i$. This is the i in the theorem. Note, we have

$$v(T \cup i \cup j) - v(T \cup i) > v(T \cup j) - v(T) \tag{1}$$

Since v satisfies the non-redundancy condition, there exists a feasible allocation u which tightens T. For any u, define the family of tight subsets of T as $\mathcal{F} = \{Z \subseteq T : u(Z) = v(Z)\}$. Note that $\mathcal{F}$ is nonempty since $T \in \mathcal{F}$. Choose u so that $|\mathcal{F}|$ is minimum. Define $u(j) := v(T \cup j) - v(T)$ and $u(i) := v(T \cup i) - v(T)$. This completes the definition of u of the theorem.

We now prove feasibility of u and the properties 1,2,3 in the statement of the theorem. We need the following structural facts which we prove later.

Lemma 2. *$\mathcal{F}$ is closed under taking complements, that is, if $Z \in \mathcal{F}$, so is $T \setminus Z$*

Lemma 3. *Let v restricted to a set X be submodular. The set of tight subsets of X are closed under taking unions and intersections.*

Lemma 4. *Union of disjoint tight sets is tight.*

We first show that u is feasible. Lemma 5 shows the feasibility for sets containing either i or j. Lemma 6 shows the feasibility for sets containing both, and in fact proves Property 2. This is sufficient byb definition. Property 1 of the theorem follows directly from definition of u. We prove property 3 after these two lemmas.

Lemma 5. *u is feasible over $T \cup i$ and $T \cup j$. In fact, if $Z \cup i$ or $Z \cup j$ is tight, then so is Z.*

Proof. Pick any subset $Z \subseteq T$. We get $u(Z \cup i) = u(Z) + v(T \cup i) - v(T)$. Since v restricted to $T \cup i$ is submodular, we get $u(Z \cup i) \leq u(Z) + v(Z \cup i) - v(Z) \leq v(Z \cup i)$. Thus u is feasible over $T \cup i$. Also, if $Z \cup i$ were tight, we would have $u(Z) = v(Z)$ implying Z were tight.

Lemma 6. *For all subsets $Z \subseteq T$, $u(Z \cup i \cup j) < v(Z \cup i \cup j)$.*

Proof. Pick any set Z. Note that

$$u(Z \cup i \cup j) = u(Z) + v(T \cup i) - v(T) + v(T \cup j) - v(T) \tag{2}$$

Note that the union of sets $(Z \cup i \cup j) \cup (T \setminus Z) = T \cup i \cup j$. Thus by the complement free condition of v, we get

$$v(Z \cup i \cup j) \geq v(T \cup i \cup j) - v(T \setminus Z)$$

Two cases arise. Suppose $Z \in \mathcal{F}$, that is, Z is tight. We take care of the other case later. Then, by Lemma 2, $T \setminus Z$ is also tight. Thus, $v(Z) + v(T \setminus Z) = v(T)$ since all the sets are tight. Putting this in above equation and applying Equation 1 we get $v(Z \cup i \cup j) \geq v(T \cup i \cup j) - v(T) + v(Z) > v(T \cup i) + v(T \cup j) - v(T) - v(T) + v(Z) \geq u(Z \cup i \cup j)$ from equation 2.

Now suppose $Z \notin F$, that is, $u(Z) < v(Z)$. Thus Equation 2 implies

$$u(Z \cup i \cup j) < v(Z) + v(T \cup i) - v(T) + v(T \cup j) - v(T)$$

By submodularity of $T \cup i$, we get $v(T \cup i) - v(T) \leq v(Z \cup i) - v(Z)$. Also, by choice of T to be the subset of $T \cup i$ minimizing $v(T \cup j) - v(T)$, we get $v(T \cup j) - v(T) \leq v(Z \cup i \cup j) - v(Z \cup i)$. Plugging this in the equation above proves the lemma.

To prove property 3, we make a few definitions. Analogous to $\mathcal{F}$, define $\mathcal{F}_i := \{Z \subseteq T : u(Z \cup i) = v(Z \cup i)\}$. Similarly define $\mathcal{F}_j$. By Lemma 5, all sets in $\mathcal{F}_i$ and $\mathcal{F}_j$ are tight. Property 3 is implied by $\bigcap_{Z \in \mathcal{F}_i, \mathcal{F}_j} Z$ contains an element l with $u(l) > 0$.

Lemma 7 shows no two sets in $\mathcal{F}_i$ or $\mathcal{F}_j$ are disjoint. By Lemma 3, this implies the tight sets $T_i := \bigcap_{Z \in \mathcal{F}_i} Z$ (similarly T_j) are non-empty. Lemma 8 shows that T_i and T_j are not disjoint and in fact $v(T_i \cap T_j) > 0$ which by tightness of $T_i \cap T_j$ proves implies existence of l.

Lemma 7. *No two sets in $\mathcal{F}_i$ or $\mathcal{F}_j$ are disjoint.*

Proof. We prove for $\mathcal{F}_j$, that for $\mathcal{F}_i$ is similar. Suppose there existed $A, B \in \mathcal{F}_j$ disjoint. Since $A \cup j$ and $B \cup j$ are tight, and v is submodular when restricted to $T \cup j$, we get the intersection of $A \cup j$ and $B \cup j$, the set j, is tight. Since $T \cup i$ is tight by Condition 1, we get $T \cup i \cup j$ is tight, contradicting Condition 2, which is already proven.

Lemma 8. $v(T_i \cap T_j) > 0$

Proof. Since T_i and T_j are tight, so is $T_i \cup T_j$ and by Lemma 2, so is $T \setminus (T_i \cup T_j)$. If T_i and T_j were disjoint, then $T \cup i \cup j$ is the disjoint union of the tight sets $(T_i \cup i), (T_j \cup j)$ and $T \setminus (T_i \cup T_j)$. By lemma 4, $T \cup i \cup j$ is tight contradicting Condition 2. Note that we cannot use Lemma 3 as v is *not* submodular when restricted to $T \cup i \cup j$, and need disjoint condition.

□

4.1 Proofs of facts used in Theorem 2

Proof of Lemma 3 : Let $A, B \subset X$ be two tight subsets of X. Since u is a feasible allocation, Thus we get

$$u(A) + u(B) = u(A \cup B) + u(A \cap B)$$

$$\leq v(A \cup B) + v(A \cap B) \leq v(A) + v(B) = u(A) + u(B)$$

implying equality throughout. In particular, $u(A \cap B) = v(A \cap B)$ and $u(A \cup B) = v(A \cup B)$. □

Proof of Lemma 4 : For any disjoint sets A, B, $u(A \cup B) = u(A) + u(B)$ and thus $v(A \cup B) \leq v(A) + v(B) = u(A) + u(B) = u(A \cup B) \leq v(A \cup B)$ where the first inequality follows from complement-free condition on v. □

Proof of Lemma 2 : Suppose $T \setminus Z$ is not tight. We modify u so that no new set becomes tight and Z also becomes untight. This contradicts the minimality of $\mathcal{F}$. Call $T \setminus z$ as X.

Pick an element $x \in X$. Let A be the *smallest* tight set containing x. This is defined since T contains x. We might assume x is picked so that A intersects Z. If no such x existed, then X is a union of tight sets and we are done by Lemma 3.

Denote $A \cap Z$ by Y. Let $y \in Y$ be with $u(z) > 0$. We prove such a y exists shortly. Since A was the smallest tight set containing x, all tight sets containing x also contains A. Therefore, modifying u to give suitable small more utility to x and exactly that less to y renders it feasible and leaves both Z and $T \setminus Z$ untight.

To see the existence of $y \in Y$ with $u(y) > 0$, note that if not then we get $u(Y) = 0$. Thus $v(A) = u(A) = u(A \cap X) \leq v(A \cap X) \leq v(A)$, implying $A \cap X \subsetneq A$ is also tight. Thus contradicts the minimality of A. □

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