Approximate Market Equilibrium for Near Gross Substitutes

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Abstract

The property of Weak Gross Substitutibility (WGS) of goods in a market has been found to be conducive to efficient algorithms for finding equilibria. In this paper, we give a natural definition of a δ-approximate WGS property, and show that the auction algorithm of [GK04, GKV04] can be extended to give an $(\epsilon + \delta)$-approximate equilibrium for markets with this property.

1 Introduction

The computational complexity of finding a market equilibrium has recieved a lot of interest lately [DPS02, DPSV02, Jai04, DV04, GK04, CSVY06, CMV05] (also see [CPV04] for a survey). A key property that has been used in designing some of these algorithms is that of Weak Gross Substitutibility (WGS). A market satisfies WGS if an increase in price of one good does not lead to a decrease in demand of any other good. Markets with WGS have been well studied in both the economics and the algorithmic game theory literature. It has been shown, for instance, [ABH59] that the *tatonnement* process converges for all markets satisfying WGS.

In this paper we extend the definition of WGS to approximate WGS and design algorithms for the same. We formulate the following alternate definition of WGS: the monetary demand for any good, which is the demand for the good times the price, is a decreasing function of its price, given that all the other prices are fixed. Our definition of an approximate WGS now follows from bounding the increase in the monetary demand for a good with an increase in its price.

2 Preliminaries

In this section, we define the Fisher market model which we refer to throughout this paper. Consider a market with n buyers and m goods. The goods are assumed to be perfectly divisible, and w.l.o.g. a unit amount of each good is available as supply. Each buyer i has an intial endowment e_i of money, and utility functions u_{ij}: $u_{ij}(x_{ij})$ gives the utility gained by her for having bought (consumed) x_{ij} units of good j. Given prices $P = (p_1, ..., p_m)$, a buyer uses her money to buy a bundle of goods $X_i = (x_{i1}, ..., x_{im})$, called the *demand vector* for buyer i, such that her total utility $U_i(X_i) = \sum_j u_{ij}(x_{ij})$ is maximized, subject to the budget constaint: $\sum_j x_{ij} p_j \le e_i$. Equilibrium prices are such that the market clears, that is, total demand for every good is equal to the supply: for all goods j, $\sum_i x_{ij} = 1$.

Let v_{ij} be $\frac{dU_i}{dx_{ij}}$. We assume that for all i and j, U_i is non-negative, non-decreasing, differentiable and concave. These constraints translate into v_{ij} being non-negative, non-increasing and well defined. From the KKT conditions on the buyers' optimization program, it follows that $v_{ij}(x_{ij})/p_j$ is equalized over all goods for which $x_{ij} > 0$. So for any optimal bundle of goods X_i for buyer i, there exists α such that $x_{ij} > 0 \Rightarrow \frac{v_{ij}(x_{ij})}{p_j} = \alpha$, and $x_{ij} = 0 \Rightarrow \frac{v_{ij}(x_{ij})}{p_j} \le \alpha$.

Definition 2.1 *For any $\epsilon > 0$, a price vector P is an ϵ-approximate market equilibrium if each buyer can be allocated a bundle X_i such that $U_i(X_i)$ is at least the optimal utility times $(1-\epsilon)$ and* *the market clears exactly.*

Definition 2.2 *A market satisfies WGS if for two price vectors P and P', such that $p_j = p'_j$ for all goods $j \neq k$ and $p_k < p'_k$, the demand of good j at P' is at least its demand at P.*

[GKV04] gave the following equivalent condition for WGS, when the utilities are separable.

Lemma 2.3 *([GKV04]) A market satsfies WGS if and only if the function $x_{ij}v_{ij}(x_{ij})$ is non-decreasing for all i and j.*

3 Extending WGS

3.1 An alternate definition of WGS

Let the monetary demand for a good be its demand times its price. We motivate our alternate definition of WGS by analysing how the monetary demand for a good changes as its price goes up.

Consider a buyer i who has an optimal bundle of goods $(x_{i1}, ..., x_{im})$. For any good j such that $x_{ij} > 0$, $\frac{v_{ij}(x_{ij})}{p_j} = \alpha$. Now suppose price of good k is driven up to $p'_k > p_k$, rest of the prices being unchanged. In this case, clearly the current allocation no more represents an optimal bundle for the buyer. We now describe a way for the buyer to adjust her allocation in order to attain the new optimum. This is done in two stages.

In the first stage, she sells some of good k to equalize the marginal rate of utility of good k with that of other goods. Let x'_{ik} be such that $v_{ik}(x'_{ik}) = \alpha p'_k$. The difference between the values of the new and original holdings is $x'_{ik}p'_k - x_{ik}p_k$. This is the amount of money she will have to pay as a result of the increase in price. From Lemma 2.3,

$$\begin{aligned} x'_{ij}p'_j - x_{ij}p_j &= \frac{1}{\alpha}(x'_{ij}v_{ij}(x'_{ij}) - x_{ij}v_{ij}(x_{ij})) \\ &\leq 0 \end{aligned}$$

This means that she has some money left over at the end of the first stage. In the second stage, she splits the left over money among all goods in such a way that $\frac{v_{ij}(x_{ij})}{p_j}$ remains the same for all goods with $x_{ij} > 0$. Two things are worth noting: the monetary demand for good k and the value of α both decrease as a result. This leads us to an alternate formulation of WGS.

Lemma 3.1 *A market satisfies WGS if for two price vectors P and P', such that $p_j = p'_j$ for all goods $j \neq k$ and $p_k < p'_k$, the monetary demand for good k at P' is smaller than that at P.*

Proof Suppose that the market is WGS. Let the demand at P and P' be X and X' respectively. Then WGS implies that $x'_j \geq x_j$ for all $j \neq k$. Therefore $p'_jx'_j \geq p_jx_j$ for all $j \neq k$. Since $\sum_j p'_jx'_j = \sum_j p_jx_j$, it follows that $p'_kx'_k \leq p_kx_k$.

Now assume that $p'_kx'_k \leq p_kx_k$. We need to prove that $x'_j \geq x_j$. We can ignore those j for which $x_j = 0$. Again, since $\sum_j p'_jx'_j = \sum_j p_jx_j$, there exists some $j \neq k$ such that $p'_jx'_j \geq p_jx_j$, and in turn $x'_j \geq x_j$. Therefore $\alpha' = \frac{v_ij(x'_j)}{p'_j} \leq \frac{v_ij(x_j)}{p_j} = \alpha$. Hence for all $j \neq k$ such that $x_j > 0, x'_j \geq x_j$. ∎

Elasticity of demand: This formulation is related to the *Price Elasticity of Demand* (see [SN92] for details) defined as

$$E_d = \frac{\text{\% change in demand}}{\text{\% change in price}} \tag{1}$$

Elasticity is a widely used measure of responsiveness of the demand to change in prices. Our alternate definition of WGS simply states that $E_d > 1$.

3.2 Approximate-WGS utility functions

We have seen how WGS can be interpreted from the demand perspective as well as the revenue perspective. Extending the revenue interpretation from Lemma 3.1, we say that a market satisfies δ-approximate WGS if increasing the price of a good does not cause its monetary demand to increase by more than a factor of $(1+\delta)$.

Definition 3.2 *For any $\delta \geq 0$, a market satisfies δ-approximate WGS if for two price vectors P and P', such that $p_j = p'_j$ for all goods $j \neq k$ and $p_k < p'_k$, the monetary demand for good k at P' is at most $(1+\delta)$ times that at P.*

In the next section, we will prove that this definition allows us to design efficient approximation algorithms for these markets. Henceforth we will refer to definition 3.2 as δ-approximate weak gross substitutability.

The above definition gives the following necessary condition on a δ-approximate WGS market.

Lemma 3.3 *If a market satisfies δ-approximate WGS, then*

$$\forall i,j, \forall x > x' \quad \Rightarrow \quad x v_{ij}(x) \geq \frac{x' v_{ij}(x')}{(1+\delta)}$$

Proof Let P be a price vector so that the demand of buyer i for good j at these prices is x. Increase p_j to get a price vector P' so that the demand of buyer i for good j at P' is x'. Let $\alpha = \frac{v_{ij}(x)}{p_j}$ and $\alpha' = \frac{v_{ij}(x')}{p'_j}$. Then as in the prooof of Lemma 3.1, $\alpha' \leq \alpha$. And by definition, $p'_j x' \leq p_j x(1+\delta)$, which implies $\frac{x' v_{ij}(x')}{\alpha'} \leq \frac{x v_{ij}(x)(1+\delta)}{\alpha}$. Combining the two inequalities, we get that $x' v_{ij}(x') \leq x v_{ij}(x)(1+\delta)$. ∎

4 Auction Algorithm

In this section we show that by slightly modifying the auction algorithm in [GKV04], we can compute an $(\epsilon+\delta)$-approximate equilibrium for a market exhibiting δ-approximate WGS property. This result shows that WGS is not a hard threshold: Markets do not suddenly become intractable if they slightly violate the WGS property.

We will use the auction algorithm of [GKV04] as the starting point. This algorithm computes ϵ-approximate equilibrium when the market satisfies WGS. An outline of the auction algorithm is as follows:

- **Ascending prices**: Prices start out at suitably low values and are raised in multiplicative steps. At any stage, some buyers may have an allocation of good j at price p_j, where as others may have bought the same good at price $\frac{p_j}{(1+\epsilon)}$. Buyer i's holding of good j at price p_j is denoted by h_{ij} and that at price $p_j/(1+\epsilon)$ is denoted by y_{ij}. Total allocation of good j to buyer i is $x_{ij} = h_{ij} + y_{ij}$.

- **Decreasing surplus**: A buyer's surplus is the money she hasn't spent. It is denoted by $r_i = e_i - \sum_j h_{ij} p_j - \sum_j y_{ij} p_j/(1+\epsilon)$. Each buyer exhausts her surplus by buying goods at price p_j from others whose allocation of good j is at price $p_j/(1+\epsilon)$. If no other buyer has the good at lower price, the price is raised from p_j to $(1+\epsilon)p_j$. Finally due to rising prices, total surplus of all buyers $r = \sum_i r_i$ approaches zero.

- **Buy-back**: Suppose buyer i had x_{ij} amount of good j with bang per buck $\alpha_{ij} = \frac{v_{ij}(x_{ij})}{p_j}$. Suppose now that other competing buyers raise the price p_j to p'_j, reducing x_{ij} to x'_{ij}. If $\frac{v_{ij}(x'_{ij})}{p'_j} \geq \alpha_{ij}$, then before buying other desirable goods, buyer i, buys back some of good j, until the rate of utility gain per dollar for good j returns to α_{ij}.

- **Near-Optimality**: ϵ-approximate optimality of the partial bundle of goods is maintained for each buyer throughout the algorithm. Therefore, when the total surplus tends to zero, the current price and allocation vectors represent ϵ-approximate equilibrium.

We require only slight modification in the auction algorithm, in order to get it working for δ-approximate WGS markets. The pseudocode of the modified algorithm can be found in figure 1. The modification to original algorithm from [GKV04] appears in `algorithm main`.

Figure 1: Modified auction algorithm

```
procedure initialize
      Set initial prices as:  $\forall j,\ p_j = v_{1j}(a_j)e_1/(\sum_j a_j v_{1j}(a_j))$.
      Allocate entire quantity of all the goods to buyer 1.  Consequently, we have:
            1.  $\forall j$ :  $\alpha_{1j} = (\sum_j a_j v_{1j}(a_j))/e_1$
            2.  $r_1 = 0$
end procedure initialize

algorithm main
      initialize
      while $\exists i$ :  $r_i > \epsilon e_i$
            while $(r_i > 0)$ and $(\exists j$ :  $(1+\delta)\alpha_{ij} p_j < v_{ij}(x_{ij}))$
                  if $\exists k$ :  $y_{kj} > 0$ then outbid($i$, $k$, $j$, $(1+\delta)\alpha_{ij}$)
                  else raise_price($j$)
            while $(r_i > 0)$ and $(\exists j$ :  $\alpha_{ij} p_j < v_{ij}(x_{ij}))$
                  if $\exists k$ :  $y_{kj} > 0$ then outbid($i$, $k$, $j$, $\alpha_{ij}$)
                  else raise_price($j$)
            $j = \arg\max_q \alpha_{iq}$
            if $\exists k$ :  $y_{kj} > 0$
                  outbid($i$, $k$, $j$, $\alpha_{ij}/(1+\epsilon)$)
                  $\alpha_{ij} = v_{ij}(x_{ij})/p_j$
            else raise_price($j$)
end algorithm main

procedure raise_price($j$)
      $\forall i$ :  $y_{ij} = h_{ij}$;  $h_{ij} = 0$
      $p_j = (1+\epsilon)p_j$
end procedure raise_price

procedure outbid($i$, $k$, $j$, $\alpha$)
      Transfer good $j$ from buyer $k$ to buyer $i$ until one of the following events happen:
            1.  Buyer $i$'s surplus reduces to zero.
            2.  Buyer $k$ has no more of good $j$ left.
            3.  Value of $v_{ij}(x_{ij})/p_j$ drops to $\alpha$.
end procedure outbid
```

We have modified *buy back* step mentioned above, and split it into two rounds: (Let X'_i and P' denote the allocation and price vectors at the begining of each round)

1. **First round**: For each good j such that $\frac{v_{ij}(x'_{ij})}{p'_j} > (1+\delta)\alpha_{ij}$, buyer i buys it back to an amount x^*_{ij} such that $(1+\delta)\alpha_{ij} = \frac{v_{ij}(x^*_{ij})}{p^*_j}$. (As opposed to buy back until $\alpha_{ij} = \frac{v_{ij}(x^*_{ij})}{p^*_j}$ in the original algorithm). If buyer i has to raise price of good j in this process, p^*_j may be strictly higher than p'_j. As we shall see, this ensures that the buyer does not spend more on any good than she originally had when the value of α_{ij} was set. Therefore, the buy-back **is** possible for all goods.

2. **Second round**: For each good j such that $\frac{v_{ij}(x'_{ij})}{p'_j} > \alpha_{ij}$, buyer i buys it back to an amount x^*_{ij} such that $\alpha_{ij} = \frac{v_{ij}(x^*_{ij})}{p^*_j}$, until she has surplus money left. This round is identical to the buy-back step in the original algorithm.

Lemma 4.1 *First buy-back round finishes with each good j considered having $\alpha_{ij} = \frac{v_{ij}(x^*_{ij})}{(1+\delta)p^*_j}$.*

Proof Consider a situation when buyer i has x'_{ij} amount of good j when her turn arrives to spend her surplus. Let p'_j be the current price and $\alpha_{ij} < \frac{v_{ij}(x'_{ij})}{(1+\delta)p'_j}$. Let x^*_{ij} be the amount and p^*_j be the price such that

$$x^*_{ij} \leq x'_{ij} < x_{ij} \qquad \text{AND} \qquad p^*_j \geq p'_j > p_j \qquad \text{AND} \qquad \alpha_{ij} = \frac{v_{ij}(x^*_{ij})}{(1+\delta)p^*_j}$$

where x_{ij} and p_j is the endowment and price of good j respectively when α_{ij} was set, *i.e.* $\alpha_{ij} = \frac{v_{ij}(x_{ij})}{p_j}$. Then using property of the market from lemma 3.3, we get:

$$\begin{aligned} \frac{v_{ij}(x^*_{ij})}{(1+\delta)p^*_j} = \alpha_{ij} &= \frac{v_{ij}(x_{ij})}{p_j} \geq \frac{x^*_{ij} v_{ij}(x^*_{ij})}{(1+\delta)x_{ij}p_j} \\ \Rightarrow x_{ij}p_j &\geq x^*_{ij}p^*_j \end{aligned}$$

The above equation certifies that the value of good j held by buyer i at the end of the first buy-back round is at most the value of her holding at original price p_j. When competing buyers reduced x_{ij} to x'_{ij}, the value was returned to buyer i in dollars. The above equation says that she can safely buy back upto x^*_{ij} of good j, using up **only** the surplus value returned to her for good j. The same is true for all such goods, hence buyer i can buy all goods considered in the first buy-back round upto x^*_{ij} such that $\alpha_{ij} = \frac{v_{ij}(x^*_{ij})}{(1+\delta)p^*_j}$. ∎

Correctness and convergence of the algorithm can be proved along the lines of [GKV04], by showing that the algorithm maintains following invariants at the start of each iteration of the outer *while* loop in `procedure main`:

I1:	$\forall j,$	$\sum_i x_{ij} = a_j$	**I4**:	$\forall i, j,$	$x_{ij} > 0 \Rightarrow \frac{(1+\epsilon)v_{ij}(x_{ij})}{p_j} \geq \alpha_{ij}$
I2:	$\forall i,$	$\sum_j x_{ij}p_j \leq e_i$	**I5**:	$\forall j,$	p_j does not fall
I3:	$\forall i, j,$	$r_i = 0 \Rightarrow \alpha_{ij} \geq \frac{v_{ij}(x_{ij})}{(1+\delta)p_j}$	**I6**:		r does not increase

Invariant I1 says that all goods are fully sold at any stage. Invariant I2 conveys the fact the buyers never exceed their budget — the initial endowment. Invariants I3 and I4 together guarantee the optimality of the bundle each buyer has after she has exhausted her surplus. Only I3 is different from the earlier auction algorithms. By invariants I3 and I4, as well as the manner in which α_{ij}'s are modified in the algorithm, we have the following at the termination:

$$\begin{aligned} \forall i, j \qquad & \frac{(1+\epsilon)v_{ij}(x_{ij})}{p_j} \geq \alpha_{ij} \geq \frac{v_{ij}(x_{ij})}{(1+\delta)p_j} \\ \text{For each buyer } i \text{ and goods } j \text{ and } k \qquad & \alpha_{ij} \leq (1+\epsilon)\alpha_{ik} \end{aligned}$$

Above constraints imply that all the bang-per-buck values for a buyer $\left(\frac{v_{ij}(x_{ij})}{p_j}\right)$ are within a factor $(1+\epsilon)^2(1+\delta)$ of each other at termination. Therefore, the modified algorithm terminates with $(\epsilon+\delta)$-approximate equilibrium, ignoring the higher order terms. The analysis of the running time is similar to that in [GKV04]:

Lemma 4.2 *If bidding is organized in rounds, i.e. if each buyer is chosen once in a round to exhaust his surplus in `procedure main`, the total unspent surplus money $r = \sum_i r_i$ decreases by a factor of $(1+\epsilon)$.*

5 Open Problems

Our result holds for separable utility functions. Clearly, definition 3.2 makes sense in the non-separable setting as well. An important open problem therefore is to devise an algorithm that finds approximate equilibrium for non-separable utility markets. Alternatively, it will be interesting to see if other algorithms that solve WGS markets extend to approximate-WGS markets.

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